Minimum Cycle Basis

Algorithms & Applications

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Abstract

We consider the problem of computing a minimum cycle basis of an undirected edge-weighted graph G with m edges and n vertices. In this problem, a $\{0,1\}$ incidence vector is associated with each cycle and the vector space over $\mathbb{F}_2$ generated by these vectors is the cycle space of G. A set of cycles is called a cycle basis of G if it forms a basis of its cycle space. A cycle basis where the sum of the weights of its cycles is minimum is called a minimum cycle basis of G. Minimum cycle bases are useful in a number of contexts, e.g., the analysis of electrical networks, structural engineering, and chemistry.

We present an $O(m^2n + mn^2 \log n)$ algorithm to compute a minimum cycle basis. The previously best known running time to compute a minimum cycle basis was $O(m^\omega n)$, where ω is the exponent of matrix multiplication. It is presently known that $\omega < 2.376$. When the edge weights are integers, we give an $O(m^2n)$ algorithm. For unweighted graphs which are reasonably dense, our algorithm runs in $O(m^\omega)$ time.

Additionally, we design approximation algorithms for the minimum cycle basis problem. For any $\epsilon > 0$ we design a fast $(1+\epsilon)$-approximation algorithm by using approximate shortest cycles computations. Moreover, for any integer $k \geq 1$ we present two constant factor approximate algorithms with approximation factor $2k-1$. One of them has an expected running time of $O(kmn^{1+2/k} + mn^{(1+1/k)(\omega-1)})$ while the other has a deterministic $O(n^{3+2/k})$ running time. For sufficiently dense graphs these running times are $o(m^\omega)$. For special cases of graphs like geometric or planar graphs we present even faster approximation algorithms. Our techniques extend to the directed minimum cycle basis problem.

We also study the minimum cycle basis problem from a practical perspective. We describe how to efficiently implement de Pina's $O(m^3 + mn^2 \log n)$ algorithm. We develop several heuristics which decrease the running time

considerably. We also present an $O(m^2n^2)$ algorithm which is obtained by combining the two fundamentally different approaches which have been used so far to compute a minimum cycle basis. Furthermore, an experimental comparison between several algorithms is presented.

Finally, we study the minimum cycle basis of a nearest neighbor graph which is defined on a point sample of a surface in $\mathbb{R}^3$. We show that, under certain sampling conditions, the minimum cycle basis encodes topological information about the surface and yields bases for the trivial and non-trivial loops of the surface. We validate our results by experiments.

Kurzzusammenfassung

Wir betrachten das Problem, eine minimale Kreisbasis eines ungerichteten, Kanten-gewichteten Graphen G mit m Kanten und n Knoten zu berechnen. Wir präsentieren einen Algorithmus mit Laufzeit $O(m^2n + mn^2 \log n)$ um eine solche minimale Kreisbasis zu berechnen. Weiterhin entwickeln wir Approximationsalgorithmen für das Minimale Kreisbasen Problem. Für jedes $\epsilon > 0$ entwickeln wir einen schnellen $(1 + \epsilon)$-Approximations Algorithmus. Außerdem, präsentieren wir für jede ganze Zahl $k \geq 1$ zwei Approximationsalgorithmen, die beide einen Approximationsfaktor von $2k - 1$ haben. Der eine hat erwartete Laufzeit $O(kmn^{1+2/k} + mn^{(1+1/k)(\omega-1)})$ und der andere Laufzeit $O(n^{3+2/k})$. Unsere Methoden sind auch auf gerichtete Graphen anwendbar.

Wir untersuchen das Minimale Kreisbasis Problem ebenfalls aus einer praktischen Perspektive. Wir entwickeln verschiedene Heuristiken, die die Laufzeit beträchtlich verbessern. Weiterhin vergleichen wir verschiedene Algorithmen anhand von Experimenten.

Schließlich untersuchen wir die minimale Kreisbasis eines „nearest neighbor“ Graphen, der auf eine Stichprobenmenge einer Oberfläche im $\mathbb{R}^3$ definiert wird.

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Chapter 1

Introduction

It is well known that Euler [31], with his solution of the Königsberg bridge problem, laid the foundation for the theory of graphs. However, the first fundamental application of graph theory to a problem in physical science did not arise till 1847, when Kirchhoff [60] developed the theory of trees for its application in the study of electrical networks. In the study of such networks, Kirchhoff's laws play a fundamental role. These laws specify the relationships among the voltage variables as well as those among the current variables of a network. For a given electrical network, these relationships do not depend on the nature of the elements used; rather, they depend only on the way the various elements are connected or, in other words, on the graph of the network. In fact, the cycles and cutsets of the graph of a network completely specify the equations describing Kirchhoff's voltage and current laws. The question then arises whether every cycle and every cutset of a network are necessary to specify these equations. Answering this and other related questions require a thorough study of the properties of cycles, cutsets, and trees of a graph. This explains the role of graph theory as an important analytical tool in the study of electrical networks [74].

It does not take long to discover that a connection can be built between graph theory and algebra. Viewing graphs in an algebraic fashion, we can define several vector spaces on a graph. Since edges carry most of the structure of a graph, we are mostly concerned with the edge space, the vector space formed by the edge sets of a graph. One of the most important subspaces of the edge space is the cycle space, our main concern in this thesis. The cycle space is the vector space generated by all cycles of a graph. For undirected graphs this vector space is defined over the 2-element field $\mathbb{F}_2 = \{0, 1\}$.

The cycle space reveals useful structural information about a graph. Thus, understanding the structure and the properties of the cycle space is essential in order to understand the properties of the underlying graph. A basis of the cycle space is a maximal set of linearly independent cycles which can generate any other cycle of the graph. It is rather common in computer science to associate combinatorial objects with weight functions. Given a graph $G = (V, E)$, call $w : E \mapsto \mathbb{R}_{\geq 0}$ a non-negative weight function on the edges of G. If such a weight function is not explicitly mentioned it can be safely assumed that it is the uniform weight function. Such a weight function provides an ordering on several combinatorial structures of a graph. Given a cycle basis, its weight is defined as the sum of the weights of its cycles. The weight of a cycle is the sum of the weights of its edges. Thus, an edge weight function defines an ordering on the cycle bases of a graph.

Constructing a cycle basis of a graph is a rather easy combinatorial problem. Given any spanning tree T of a graph $G = (V, E)$ consider the following set of cycles. For each edge $e = (u, v) \in E \setminus T$ the cycle formed by e and the unique path from u to v in T. All these cycles form a cycle basis, called a fundamental cycle basis. Thus, constructing a cycle basis is a linear time process assuming that we encode it appropriately and do not explicitly output the cycles[1]. Imposing extra conditions, however, on the particular cycle basis makes the problem considerably harder. Consider the question of finding the spanning tree T of a graph such that the induced fundamental cycle basis has minimum weight. This problem is called the fundamental minimum cycle basis problem. Deo et al. [21] proved that it is NP-complete.

Fortunately, if we do not restrict ourselves to fundamental cycle bases the problem becomes easier and polynomial time algorithms can be derived. The problem of finding a cycle basis with minimum weight is denoted as the minimum cycle basis problem. Since any fundamental cycle basis is a cycle basis but not vice-versa, a minimum cycle basis can have less weight than a minimum fundamental cycle basis.

The problem of finding low-cost cycle bases, or in other words sparse cycle bases, has been considered in the literature multiple times, see for example [73, 82, 54, 61]. Horton [52] was the first to present a polynomial time algorithm for finding a minimum cycle basis in a non-negative edge weighted graph. Later, Hartvigsen and Mardon [47] studied the structure of minimum cycle bases and characterized graphs whose short cycles[2] form a minimum cycle basis. They essentially characterized those graphs

[1] The size of a cycle basis can be superlinear on the size of a graph.

[2] A cycle C is considered a short cycle if it is the shortest cycle through one of its edges.

for which an algorithm of Stepanec [73] always produces a minimum cycle basis. Hartvigsen [46] also introduced another vector space associated with the paths and the cycles of a graph, the *U-space*. Hartvigsen extended Horton's approach to compute a minimum weight basis for this space as well. Hartvigsen and Mardon [48] also studied the minimum cycle basis problem when restricted to planar graphs and designed an $O(n^2 \log n)$ time algorithm.

The first improvement over Horton's algorithm was obtained by de Pina in his PhD thesis [20], where an $O(m^3 + mn^2 \log n)$ algorithm is presented. The approach used by de Pina was fundamentally different by the one of Horton. Recently, Golynski and Horton [40] observed that the original approach from Horton could be improved by using fast matrix multiplication, obtaining an $O(m^\omega n)$ algorithm. It is presently known [17] that $\omega < 2.376$. Finally, Berger et al. [8] presented another $O(m^3 + mn^2 \log n)$ algorithm using similar techniques as de Pina.

The applicability and importance of minimum cycle bases is not restricted to solving or understanding problems in electrical networks. The problem has many more applications in many diverse areas of science. One such application is for example structural engineering [14], while chemistry and biochemistry [39] is another. Recently, Tewari et al. [75] used a cycle basis to do surface reconstruction from a point cloud sampled from a genus one smooth manifold. In many of the above algorithms, the amount of work is dependent on the cycle basis chosen. A basis with shorter cycles may speed up the algorithm.

The minimum cycle basis problem can also be considered in the case of directed graphs. A cycle in a directed graph is a cycle in the underlying undirected graph with edges traversed in both directions. A $\{-1, 0, 1\}$ edge incidence vector is associated with each cycle: edges traversed by the cycle in the right direction get 1 and edges traversed in the opposite direction get -1. The main difference here is that the cycle space is generated over $\mathbb{Q}$ by the cycle vectors. In the case of directed graphs polynomial algorithms were also developed. The first polynomial time algorithm had running time $\tilde{O}(m^4 n)$ [56]. Liebchen and Rizzi [64] gave an $\tilde{O}(m^{\omega+1} n)$ deterministic algorithm. This has already been improved to $O(m^3 n + m^2 n^2 \log n)$ [45]. However, faster randomized algorithms are known like the $O(m^2 n \log n)$ Monte Carlo algorithm in [55]. The fastest algorithms for finding a minimum cycle basis in directed graphs are based on the techniques used in de Pina's thesis [20] and the ones presented in this thesis. However, several extra ideas are required in order to compensate for the extra cost of arithmetic that arises when changing the base field from $\mathbb{F}_2$ to $\mathbb{Q}$. See for example [56, 45].

For a more extensive treatment of different classes of cycle bases which can be defined on graphs we refer the interested reader to Liebchen and Rizzi [63].

Organization and Contributions

Chapter 2 contains some very basic definitions and notation about the mathematical machinery used in this thesis. This includes some graph theory, the connection of algebra and graphs, and some basic topology theory. A reader familiar with these subjects can safely skip this chapter.

Chapter 3 describes how to compute a minimum cycle basis of a weighted undirected graph in $O(m^2n + mn^2 \log n)$ time. We achieve such a running time by considering an older approach and casting it into an algebraic framework. This algebraic framework has two main parts, (a) maintaining linear independence, and (b) computing short cycles (paths). Given the algebraic framework, we design a recursive algorithm which uses fast matrix multiplication to speedup the linear independence step. The idea here is to relax an invariant of de Pina's algorithm [20], while at the same time maintain correctness. The relaxation allows us to group several steps into one bulk step which can then be performed by using some fast matrix multiplication algorithm. The dominating factor of the running time is now the shortest cycles computation. For special cases like unweighted graphs or integer weights we present faster algorithms, by using better shortest paths algorithms.

Chapter 4 goes one step further and presents approximation algorithms for the minimum cycle basis problem. Our first result is an α-approximation algorithm, for any $\alpha > 1$, obtained by using approximate shortest paths computations. We also present constant factor approximation algorithms which for sufficiently dense graphs have an $o(m^\omega)$ running time. In order to obtain such a running time we divide the cycles computation in two parts based on a $(2k-1)$-spanner computation, for any integer $k \geq 1$. In the first part we compute, very fast, a large number of cycles of a $(2k-1)$-approximate minimum cycle basis. The second part is a slower computation of the remaining cycles. The improvement in the running time is due to the fact that the remaining cycles are $O(n^{1+1/k})$. Our techniques are also applicable to the directed minimum cycle basis problem. Finally, one of our $(2k-1)$-approximation algorithms is very efficient even when implemented without fast matrix multiplication. We elaborate on this further in Chapter 5.

Our interest in the minimum cycle basis problem is not purely theoretical. Chapter 5 studies the minimum cycle basis problem from a practical viewpoint. Our first concern is to examine the applicability of the fast

matrix multiplication techniques. To this end we implemented de Pina's $O(m^3 + mn^2 \log n)$ algorithm, and developed several heuristics which improve the best case while maintaining its asymptotic behavior. One of these heuristics, by reducing the number of shortest paths computations, improves the running time dramatically. Experiments with our implementation on random graphs suggest that the dominating factor of the running time is the shortest paths computation. Note that for sparse graphs ($m \in O(n)$) the algorithm presented in Chapter 3 is also an $O(n^3 \log n)$ algorithm. Thus, our theoretical improvement for the exact minimum cycle basis computation is not useful in practice for random graphs. On the other hand, there are instances of the problem where the dominating part is the linear independence. In such cases our improvement should also have a practical importance. One such instance is studied in more detail in Chapter 6.

The observation that for certain graphs the dominating part of the running time is the shortest paths computations, naturally leads to the question of whether we can compute a minimum cycle basis with fewer shortest paths computations. All known minimum cycle basis algorithms can be divided into two main categories. The ones that follow Horton's approach and the ones that follow de Pina's approach. While Horton's approach, even when used with fast matrix multiplication, is slower than the approach in Chapter 3, there is one important property which seems rather helpful. Horton's approach performs fewer shortest path computations and spends more time ensuring linear independence. In Chapter 5 we combine the two approaches to reach a "hybrid" algorithm which performs the shortest paths computations of Horton's algorithm and ensures linear independence by using de Pina's approach. The resulting algorithm has running time $O(m^2 n^2)$. Experiments suggest that the hybrid algorithm is very efficient when applied on random dense unweighted graphs. Finally, Chapter 5 contains a comparison between several existing minimum cycle basis implementations and an experimental view of our constant factor approximation algorithms.

Chapter 6 treats the minimum cycle basis of a particular instance of graphs. Consider a compact manifold S in $\mathbb{R}^3$ and a finite set of points P in S. A common approach into treating such point samples is to first construct some neighboring graph. One such popular graph is the k-nearest neighbor graph; an undirected graph with vertex set P and an edge between two sample points a and b if b is one of the k points closest to a and a is one of the k points closest to b. Chapter 6 studies the minimum cycle basis of the k-nearest neighbor graph when S is a compact smooth manifold and P is a sufficiently dense sample.

We show that for suitably nice samples of smooth manifolds of genus g

and sufficiently large k, the k-nearest neighbor graph G_k has a cycle basis consisting only of short (= length at most $2(k+3)$) and long (= length at least $4(k+3)$) cycles. Moreover, the minimum cycle basis is such a basis and contains exactly $m-(n-1)-2g$ short cycles and $2g$ long cycles. The short cycles span the subspace of trivial loops and the long cycles form a homology basis; see Chapter 2 for a definition. Thus, the MCB of G_k reveals the genus of S and also provides a basis for the set of trivial cycles and a set of generators for the non-trivial cycles of S. We also validate our results with experiments.

We offer conclusions and some open problems in Chapter 7.

LEDA Extension Package

As a result of this thesis we have developed a LEDA [67] extension package for computing exact and approximate minimum cycle bases [65].

Publication Notes and Collaboration

The results presented in Chapter 3 are joint work with Telikepalli Kavitha, Kurt Mehlhorn, and Katarzyna Paluch. This work was published in the Conference Proceedings of the 31st International Colloquium on Automata, Languages and Programming (ICALP 2004) [58]. Chapter 4 is joint work with Telikepalli Kavitha and Kurt Mehlhorn. It has been accepted in the Conference Proceedings of the 24th International Symposium on Theoretical Aspects of Computer Science (STACS 2007) [57]. Chapter 5 is joint work with Kurt Mehlhorn. A preliminary version was published in the Conference Proceedings of the 4th International Workshop on Experimental and Efficient Algorithms (WEA 2005) [66]. A full version has been accepted for publication in the ACM Journal of Experimental Algorithmics: Selected papers from WEA'05. Finally, the results of Chapter 6 are joint work with Craig Gotsman, Kanela Kaligosi, Kurt Mehlhorn, and Evangelia Pyrga [42] and have been accepted for publication in the Computer Aided Geometric Design journal.

Chapter 2

Preliminaries

In this chapter we review several basic facts concerning the problems studied in this thesis. Reading this chapter is not necessary, in order to understand the remaining part of the thesis, as long as the reader has a basic familiarity with graph theory, the various vector spaces that can be associated with graphs, and basic topology. Throughout this thesis we assume familiarity with basic algebra. An excellent reference to the subject is [49].

2.1 Graph Theory

We use standard terminology from graph theory. In this thesis we consider finite graphs which in most of the cases are undirected. Thus, we only provide definitions for undirected graphs. For directed graphs see [18].

An *undirected graph* G is a pair (V, E), where V is a finite set, and E is a family of unordered pairs of elements of V. The elements of V are called *vertices*, and the elements of E are called *edges* of G. Sometimes the notation $V(G)$ and $E(G)$ is used, to emphasize the graph that these two sets belong to. Given an edge between two vertices $v, u \in V$, $v \neq u$ we denote this edge by (v, u) or (u, v). For an edge $e = (u, v) \in E$, u and v are called its *end-vertices* or *endpoints*. We also say that edge e is *incident* to vertices u and v. Similarly we say that vertex v is *adjacent* to vertex u. Since we assume an undirected graph, the adjacency relation is symmetric. The *degree* of a vertex in an undirected graph is the number of edges incident to it. We use the notation $\deg(v)$ to denote the degree of a vertex v.

A *path* p of *length* k from a vertex u to a vertex u' in a graph $G(V, E)$ is a sequence $\langle v_0, v_1, \ldots, v_k \rangle$ of vertices such that $u = v_0$, $u' = v_k$ and $(v_{i-1}, v_i) \in E$ for $i = 1, 2, \ldots, k$. The length of the path is the number of

edges in the path. If there is a path p from u to u', we say that u' is *reachable* from u via p. A path is *simple* if all its vertices are distinct. In an undirected graph, a path $\langle v_0, v_1, \ldots, v_k \rangle$ forms a *cycle* if $v_0 = v_k$ and $v_1, v_2, \ldots, v_k$ are distinct. A graph with no cycles is *acyclic*.

An undirected graph is *connected* if every pair of vertices is connected by a path. The *connected components* of a graph are the equivalence classes of vertices under the "is reachable from" relation. Thus, an undirected graph is connected if it has exactly one connected component. We will denote the number of connected components of graph G as $\kappa(G)$.

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Given a set $V' \subseteq V$, the subgraph of G *induced* by V' is the graph $G' = (V', E')$ where $E' = \{(u, v) \in E : u, v \in V'\}$.

The Shortest Path Problem

Let a graph $G = (V, E)$ and a weight function $w : E \mapsto \mathbb{R}$ mapping edges to real-valued weights. Such a graph is called a *weighted graph*. The weight[1] of a path $p = \langle v_0, v_1, \ldots, v_k \rangle$ is the sum of the weights of its edges, $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$. Define the *shortest path weight* from u to v as

$$\delta(u, v) = \begin{cases} min\{w(p) : p \text{ is a path from } u \text{ to } v\} & \text{if there is a path from } u \text{ to } v, \\ \infty & \text{otherwise.} \end{cases}$$

A *shortest path* from u to v is defined as any path p such that $w(p) = \delta(u, v)$.

Problem 2.1 (single source shortest paths)**.** Given a graph $G = (V, E)$ together with an edge weight function $w : E \mapsto \mathbb{R}$ find a shortest path from a given *source* vertex $s \in V$ to every vertex $v \in V$.

In the single source shortest paths problem some edges may have negative weights. If G contains no negative-weight cycles reachable from the source s, then for all $v \in V$ the shortest path weight $\delta(s, v)$ is well defined. Otherwise, if there is a negative-weight cycle on some path from s to v, define $\delta(s, v) = -\infty$.

If we are interested in knowing the shortest paths distances between all pair of vertices we have the all pairs shortest paths problem.

Problem 2.2 (all pairs shortest paths)**.** Given a graph $G = (V, E)$ and an edge weight function $w : E \mapsto \mathbb{R}$ find a shortest path between each pair of vertices $v, u \in V$.

[1] The term length is also used to denote the weight of a path.

Dijkstra's Algorithm

One of the most famous algorithms in computer science is Dijkstra's algorithm [26] for solving the single source shortest paths problem on a weighted graph $G = (V, E)$ for the case in which all weights are non-negative, that is $w(e) \geq 0$ for each edge $e = (u, v) \in E$.

Dijkstra's algorithm maintains a set S of vertices whose final shortest path weights from source $s \in V$ have already been determined. The algorithm repeatedly selects the vertex $u \in V \setminus S$ with the minimum shortest path estimate from S. Vertex u is added into S and the shortest path estimate of all vertices in $V \setminus S$ which are adjacent to u is updated to incorporate that u now belongs in S.

Let $n = |V|$ and $m = |E|$ be the cardinalities of the vertex and the edge set of G. The fastest implementation [34] of Dijkstra's algorithm requires $O(m + n \log n)$ time and linear space.

2.2 Graphs and Linear Algebra

Let $G = (V, E)$ be a graph with n vertices and m edges, say $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. The *vertex space* $\mathcal{V}(G)$ of G is the vector space over the 2 element field $\mathbb{F}_2$ of all functions $V \mapsto \mathbb{F}_2$. Every element of $\mathcal{V}(G)$ corresponds naturally to a subset of V, the set of those vertices to which it assigns a 1, and every subset of V is uniquely determined in $\mathcal{V}(G)$ by its indicator function. The sum $U + U'$ of two vertex sets $U, U' \subseteq V$ is their symmetric difference, and $U = -U$ for all $U \subseteq V$. The zero in $\mathcal{V}(G)$ is the empty (vertex) set. Since $\{\{v_1\}, \ldots, \{v_n\}\}$ is a basis of $\mathcal{V}(G)$, its *standard basis*, the dimension of $\mathcal{V}(G)$ is $\dim \mathcal{V}(G) = n$.

In the same way as above, the functions $E \mapsto \mathbb{F}_2$ form the *edge space* $\mathcal{E}(G)$ of G: its elements are the subsets of E, vector addition amounts to symmetric difference, $\emptyset \subseteq E$ is the zero, and $F = -F$ for all $F \subseteq E$. As before, $\{\{e_1\}, \ldots, \{e_m\}\}$ is the *standard basis* of $\mathcal{E}(G)$, and $\dim \mathcal{E}(G) = m$.

The edges of a graph carry most of its structure and thus we will be concerned only with its edge space. Given two edge sets $F, F' \in \mathcal{E}(G)$ and their coefficients $\lambda_1, \ldots, \lambda_m$ and $\lambda'_1, \ldots, \lambda'_m$ with respect to the standard basis, we have their *inner product* as

$$\langle F, F' \rangle := \lambda_1 \lambda'_1 + \ldots + \lambda_m \lambda'_m \in \mathbb{F}_2 \ .$$

Note that $\langle F, F' \rangle = 0$ may hold even when $F = F' \neq 0$, more precisely $\langle F, F' \rangle = 0$ if and only if F and F' have an even number of edges in common.

When $\langle a, b\rangle = 0$ we say that a and b are *orthogonal*.

Given a subspace $\mathcal{F}$ of $\mathcal{E}(G)$, we write

$$\mathcal{F}^{\perp} := \{D \in \mathcal{E}(G) \mid \langle F, D\rangle = 0 \text{ for all } F \in \mathcal{F}\} .$$

This is again a subspace and we call it the *orthogonal subspace*. The dimensions of these two subspaces are related by

$$\dim \mathcal{F} + \dim \mathcal{F}^{\perp} = m .$$

The *cycle space* $\mathcal{C}(G)$ is the subspace of $\mathcal{E}(G)$ spanned by all the cycles in G, more precisely, by their edge sets[2]. The dimension of $\mathcal{C}(G)$ is the *cyclomatic number*. The elements of $\mathcal{C}(G)$ are easily recognized by the degrees of the subgraphs they form.

Proposition 2.1. *The following are equivalent for an edge set $F \subseteq E$:*

(i) $F \in \mathcal{C}(G)$,

(ii) F *is an edge disjoint union of cycles of* G,

(iii) All vertex degrees of the graph $G(V, F)$ *are even.*

Proof. Cycles have even degrees and symmetric difference preserves this. Thus, (i)→(iii) follows by induction on the number of cycles used to generate F. The implication (iii)→(ii) follows by induction on $|F|$: if $F \neq \emptyset$ then (V, F) contains a cycle C, whose edges we delete for the induction step. The implication (ii) →(i) follows from the definition of $\mathcal{C}(G)$. □

Consider a connected graph $G = (V, E)$ and let T be a spanning tree of G. Let $c_1, c_2, \ldots, c_{m-n+1}$ denote all the edges in $E \setminus T$. We call these edges the *chords* of T. For each such chord c_i there is a unique cycle C_i which is formed by c_i and the unique path on T between the endpoints of c_i. Such cycles are the *fundamental cycles* of G with respect to T.

By definition each fundamental cycle C_i contains exactly one chord, namely c_i, which is not contained in any other fundamental cycle. Thus, no fundamental cycle can be expressed as a linear combination of other fundamental cycles. Hence, the fundamental cycles $C_1, C_2, \ldots, C_{m-n+1}$ are linearly independent. We will also show that every subgraph in the cycle

[2]For simplicity we do not normally distinguish between cycles and edge sets w.r.t the cycle space. Similarly, we view edge sets also as vectors. Since we are working over $\mathbb{F}_2$, where addition of vectors representing edge sets is the same as the symmetric difference of the edge sets, we use either $+$ or $\oplus$ to denote the addition operator.

space of G can be expressed as a linear combination of the fundamental cycles. This immediately implies that the set $\{C_1, C_2, \ldots, C_{m-n+1}\}$ is a basis of the cycle space of G.

Consider any subgraph C in the cycle space of G. Let C contain the chords $c_{i_1}, c_{i_2}, \ldots, c_{i_r}$. Let also C' be equal to $C_{i_1} \oplus C_{i_2} \oplus \ldots \oplus C_{i_r}$. C' by definition contains the chords $c_{i_1}, c_{i_2}, \ldots, c_{i_r}$ and no other chords of T. Since C also contains these chords and no others, $C \oplus C'$ contains no chords.

We now claim that $C \oplus C'$ is empty. If not, then by the preceding discussion $C \oplus C'$ contains only edges of T and thus contains no cycles. This contradicts that $C \oplus C'$ is in the cycle space. Hence, $C = C' = C_{i_1} \oplus C_{i_2} \oplus \ldots \oplus C_{i_r}$. In other words, every subgraph in the cycle space of G can be expressed as a linear combination of C_i's. Thus, we have the following theorem.

Theorem 2.1. *Let a connected graph $G = (V, E)$ with n vertices and m edges. The fundamental cycles w.r.t a spanning tree of G constitute a basis for the cycle space of G. Thus, the dimension of the cycle space of G is equal to $m - n + 1$.*

It is rather easy to see that in the case of a graph G which is not connected, the set of all fundamental cycles with respect to the chords of a spanning forest of G is a basis of the cycle space of G.

Corollary 2.2. *The dimension of the cycle space of a graph $G = (V, E)$ with n vertices, m edges, and κ connected components is equal to $m - n + \kappa$.*

The next theorem will prove to be particularly useful, especially combined with the fact that the minimum cycle basis problem can be solved by the greedy algorithm.

Theorem 2.3. *Assume $\mathbb{B}$ is a cycle basis of a graph G, C is a cycle in $\mathbb{B}$, and $C = C_1 \oplus C_2$. Then, either $\mathbb{B} \setminus \{C\} \cup \{C_1\}$ or $\mathbb{B} \setminus \{C\} \cup \{C_2\}$ is a cycle basis.*

Proof. Assume otherwise. Then, both C_1 and C_2 can be expressed as a linear combination of $\mathbb{B} \setminus \{C\}$. But $C = C_1 \oplus C_2$, and thus C can also be expressed as a linear combination of $\mathbb{B} \setminus \{C\}$. A contradiction to the fact that $\mathbb{B}$ is a basis. □

For a more detailed treatment of such topics we refer the reader to [11, 74, 25].

2.3 Topology

This section contains some basic definitions from topology; for a more thorough introduction we refer the interested reader to [68]. An introduction to homology theory can be found in [37].

2.3.1 Simplicial Complexes

Definition 2.1 (affinely independent)**.** Let $v_0, \ldots, v_n$ be $n+1$ vectors in $\mathbb{R}^d$, $n \geq 1$. They are called *affinely independent* (*a-independent*) if $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent. By convention if $n = 0$ then the vector v_0 is always a-independent.

The definition of a-independence does not in reality depend on the ordering of the v_i's. As an example consider three vectors v_0,v_1, and v_2. They are a-independent if and only if they are not collinear.

Definition 2.1 can be reformulated in the following useful form.

Definition 2.2 (affinely dependent)**.** Let $v_0, \ldots, v_n$ be vectors in $\mathbb{R}^d$. A vector v is said to be *affinely dependent* (*a-dependent*) on them if there exist real numbers $\lambda_0, \ldots, \lambda_n$ such that $\lambda_0 + \cdots + \lambda_n = 1$ and $v = \lambda_0 v_0 + \cdots + \lambda_n v_n$.

Proposition 2.2. *Let $v_0, \ldots, v_n$ be a-independent and let v be a-dependent on them. Then, there exist unique real numbers $\lambda_0, \ldots, \lambda_n$ s.t. $\sum_{i=0}^{n} \lambda_i = 1$ and $v = \sum_{i=0}^{n} \lambda_i v_i$.*

The λ's are called the *barycentric coordinates* of v w.r.t $v_0, \ldots, v_n$. We next define what a simplex is.

Definition 2.3 (simplex)**.** Let $v_0, \ldots, v_n$ be a-independent. The (closed) *simplex* with vertices $v_0, \ldots, v_n$ is the set of points a-dependent on $v_0, \ldots, v_n$ and with every barycentric coordinate ≥ 0.

Let $s_n = (v_0 \ldots v_n)$ be a simplex. A *face* of s_n is a simplex whose vertices form a (nonempty) subset of $\{v_0, \ldots, v_n\}$. If the subset is proper we say that the face is a *proper face*. If s_p is a face of s_n we write $s_p < s_n$ or $s_n > s_p$. The *boundary* of s_n is the union of the proper faces of s_n.

Definition 2.4 (simplicial complex)**.** A *simplicial complex* is a finite set K of simplexes in $\mathbb{R}^d$ with the following two properties: (a) if $s \in K$ and $t < s$ then $t \in K$, (b) if $s \in K$ and $t \in K$ then $s \cap t$ is either empty or else a face both of s and of t.

Depending on our needs we can define oriented and unoriented simplexes and simplicial complexes. See for example [37].

2.3.2 Manifolds and Simplicial Homology

A 2-manifold is a topological space in which every point has a neighborhood homeomorphic to $\mathbb{R}^2$. In this thesis only connected, compact, orientable 2-manifolds without boundary will be considered. The *genus* of a 2-manifold is the number of disjoint cycles that can be removed without disconnecting the manifold. Two connected, compact, orientable 2-manifolds without boundary are homeomorphic if and only if they have the same genus.

Let R be an arbitrary ring and M be a 2-manifold as described above. A *k-chain* is a formal linear combination of oriented *k-simplices*[3] with coefficients in the ring R. The set of k-chains forms a *chain group* $C_k(M;R)$ under addition. The *boundary operator* $\partial_k : C_k \mapsto C_{k-1}$ is a linear map taking any oriented simplex to the chain consisting of its oriented boundary facets. A k-chain is called a *k-cycle* if its boundary is empty and a *k-boundary* if it is the boundary of a $(k+1)$-cycle. Every k-boundary is a k-cycle. Let Z_k and B_k denote the subgroups of k-cycles and k-boundaries in C_k. The *k-th homology group* $H_k(M;R)$ is the quotient group Z_k/B_k. If M is an oriented 2-manifold of genus g, then $H_1(M;R) \cong R^{2g}$.

More intuitively, a *homology cycle* is a formal linear combination of oriented cycles with coefficients in R. The identity element of the homology group is the equivalence class of *separating* cycles, that is, cycles whose removal disconnects the surface. Two homology cycles are in the same homology class if one can be continuously deformed into the other via a deformation that may include splitting cycles at self-intersection points, merging intersecting pairs of cycles, or adding or deleting separating cycles. We define a *homology basis* for M to be any set of $2g$ cycles whose homology classes generate $H_1(M;R)$.

When we are dealing with unoriented simplicial complexes, we can replace the ring R by the field $\mathbb{F}_2$.

[3] In *simplicial* homology, we assume that M is a simplicial complex and build chains from its component simplices. In *singular* homology, continuous maps from the canonical k-simplex to M play the role of 'k-simplices'. These two definitions yield isomorphic homology groups for manifolds.

Chapter 3

Exact Minimum Cycle Basis

Summary

In this chapter we consider the problem of computing a minimum cycle basis in an undirected graph G with m edges and n vertices. The input is an undirected graph whose edges have non-negative weights. Graph cycles are associated with a $\{0,1\}$ incidence vector and the vector space over $\mathbb{F}_2$ generated by these vectors is the cycle space of G. A set of cycles is called a cycle basis of G if it forms a basis for its cycle space. A cycle basis where the sum of the weights of its cycles is minimum is called a minimum cycle basis of G.

We present an algebraic framework and an $O(m^2n + mn^2 \log n)$ algorithm for solving the minimum cycle basis problem. Using specialized shortest paths we also present improved time bounds for special cases like integer weights or unweighted graphs.

3.1 Introduction

Let $G = (V, E)$ be an undirected graph with m edges and n vertices. A *cycle*[†] of G is any subgraph of G where each vertex has even degree. Associated with each cycle C is an *incidence vector* x, indexed on E, where for any $e \in E$

$$x_e = \begin{cases} 1 & \text{if } e \text{ is an edge of } C, \\ 0 & \text{otherwise.} \end{cases}$$

The vector space over $\mathbb{F}_2$ generated by the incidence vectors of cycles is called the *cycle space* of G. It is well known (see Corollary 2.2) that this vector space has dimension $m - n + \kappa(G)$, where m is the number of edges of G, n is the number of vertices, and $\kappa(G)$ is the number of connected

[†] From now on we use the term *cycle* to denote either a cycle or a disjoint union of cycles (see Proposition 2.1) in the graph. The exact meaning should be clear from the context.

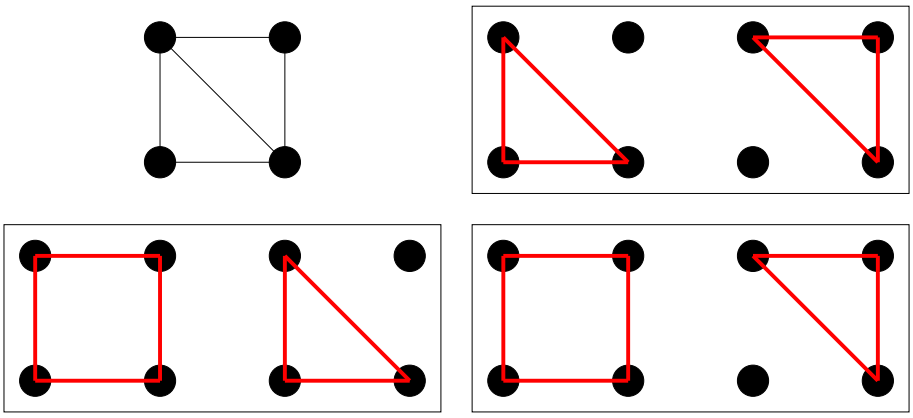


Figure 3.1: A graph with 4 vertices and 5 edges and its three possible cycle bases. Assuming uniform edge weights, two of the cycle bases have cost 7 and the minimum cycle basis has cost 6.

components of G. A maximal set of linearly independent cycles is called a *cycle basis*.

The edges of G have non-negative weights assigned to them. A cycle basis where the sum of the weights of the cycles is minimum is called a *minimum cycle basis* of G. We consider the problem of computing a minimum cycle basis of G. We also use the abbreviation MCB to refer to a minimum cycle basis. We will assume that G is connected since the minimum cycle basis of a graph is the union of the minimum cycle bases of its connected components. Thus, we will denote the dimension of the cycle space as $N = m - n + 1$.

See Figure 3.1 for an example of a graph with 4 vertices and 5 edges and the possible cycle bases. The graph has 3 cycle bases. Assuming uniform weights on the edges, two of them have cost 7 and one, the minimum cycle basis, has cost 6.

The problem of computing a minimum cycle basis has been extensively studied, both in its general setting and in special classes of graphs. Its importance lies in understanding the cyclic structure of graphs and its use as a preprocessing step in several algorithms. That is, a cycle basis is used as input for some algorithms, and using a minimum cycle basis instead of any arbitrary cycle basis usually reduces the amount of work that has to be done. Such algorithms include algorithms from diverse applications like electrical engineering [15], structural engineering [14], chemistry [39], and surface reconstruction [75]. Chapter 6 is originally motivated by such an application.

3.1.1 Algorithmic History

The first polynomial time algorithm for the minimum cycle basis problem was given by Horton [52] and had running time $O(m^3n)$. Horton's approach was to create a set M of $O(mn)$ cycles which he proved was a superset of an MCB and then extract the MCB as the shortest $m - n + 1$ linearly independent cycles from M using Gaussian elimination. De Pina [20] gave an $O(m^3+mn^2 \log n)$ algorithm to compute an MCB. The approach in [20] is different from that of Horton; de Pina's algorithm is similar to the algorithm of Padberg and Rao [69] to solve the minimum weighted T-odd cut problem.

Later on Golynski and Horton [40] observed that the shortest $m-n+1$ linearly independent cycles could be obtained from M in $O(m^\omega n)$ time using fast matrix multiplication algorithms, where ω is the exponent for matrix multiplication. It is presently known [17] that $\omega < 2.376$. The $O(m^\omega n)$ algorithm was the fastest known algorithm for the MCB problem.

For planar graphs, Hartvigsen and Mardon [48] showed that an MCB can be computed in $O(n^2 \log n)$ time. In [47] Hartvigsen and Mardon study the structure of minimum cycle bases and characterize graphs whose short cycles[1] form an MCB.

Closely related to the problem of computing an MCB is the problem of finding a minimum *fundamental cycle basis*. In this problem, given a connected graph G, we are required to find a spanning tree T of G such that the fundamental cycle basis (where each cycle is of the form: one edge from $E \setminus T$ and a path in T) is as small as possible. This problem has been shown to be NP-complete [21].

Applications and history of the problem are surveyed in Deo et al. [21], Horton [52], and de Pina [20].

3.2 An Algebraic Framework

Let T be any spanning tree in $G(V, E)$ and let $e_1, \ldots, e_N$ be the edges of $E \setminus T$ in some arbitrary but fixed order. De Pina [20] gave the combinatorial algorithm in Algorithm 3.1 to compute a minimum cycle basis in G. Before we show the correctness of de Pina's algorithm, we will cast the algorithm into an algebraic framework. The intuition behind the Algorithm 3.1 and the idea as to why it works is not clear from its combinatorial version.

A cycle in G can be viewed in terms of its incidence vector, meaning that each cycle is a vector (with 0's and 1's in its coordinates) in the space

[1] A cycle C is considered a short cycle if it is the shortest cycle through one of its edges.

Algorithm 3.1: De Pina's combinatorial algorithm.

```
Initialize S_{1,i} = {e_i} for i = 1, ..., N.
for k = 1, ..., N do
    Find a minimum weight cycle C_k with an odd number of edges in
    S_{k,k}.
    for i = k + 1, ..., N do
        if C_k has an even number of edges in S_{k,i} then
            define S_{k+1,i} = S_{k,i}
        else
            define S_{k+1,i} = S_{k,i} ⊕ S_{k,k}
        end
    end
end
Return {C_1, ..., C_N}.
```

spanned by all the edges. Here we will look such vectors restricted[2] to the coordinates indexed by $\{e_1, \ldots, e_N\}$. That is, each cycle can be represented as a vector in $\{0,1\}^N$.

Algorithm 3.2 computes the cycles of a minimum cycle basis and their *witnesses*. A witness S of a cycle C is a subset of $\{e_1, \ldots, e_N\}$ which will prove that C belongs to the minimum cycle basis. We will view these witnesses or subsets in terms of their incidence vectors over $\{e_1, \ldots, e_N\}$. Hence, both cycles and their witnesses are vectors in the space $\{0,1\}^N$.

$\langle C, S\rangle$ stands for the standard inner product of the vectors C and S. We say that a vector S is *orthogonal* to C if $\langle C, S\rangle = 0$. Since we are in the field $\mathbb{F}_2$, observe that $\langle C, S\rangle = 1$ if and only if C contains an odd number of edges of S.

The Algorithm 3.2 performs N main steps. In each step i, $1 \leq i \leq N$ a new cycle C_i is computed. In order for this new cycle C_i to be linearly independent of the set of cycles $\{C_1, \ldots, C_{i-1}\}$ previously computed, the algorithm first computes a non-zero vector S_i which is orthogonal to the cycles $C_1, \ldots, C_{i-1}$. Given such an S_i, the algorithm then computes a cycle C_i which is the shortest cycle in G such that $\langle C_i, S_i\rangle = 1$.

We first prove some simple properties of Algorithm 3.2.

[2] For a cycle C, use C to denote its incidence vector in $\{0,1\}^N$ and C^* to denote its incidence vector in $\{0,1\}^m$. Consider a set of cycles $C_1, \ldots, C_k$. Clearly, if the vectors $C_1^*, \ldots, C_k^*$ are dependent, then so are the vectors $C_1, \ldots, C_k$. Conversely, assume that $\sum_i \lambda_i C_i = 0$ for some $\lambda_i \in \{0,1\}$. Then, $C = \sum_i \lambda_i C_i^*$ contains only edges in T. Moreover, since C is a sum of cycles, each vertex has even degree with respect to C. Thus, $C = 0$ and hence linear dependence of the restricted incidence vectors implies linear dependence of the full incidence vectors. For this reason, we may restrict attention to the restricted incidence vectors when discussing questions of linear independence.

Algorithm 3.2: An algebraic framework for computing an MCB.

for $i = 1, \ldots, N$ **do**
  Let S_i denote an arbitrary non-zero vector in the subspace orthogonal to $\{C_1, C_2, \ldots, C_{i-1}\}$.
  That is, $S_i \neq \vec{0}$ satisfies: $\langle C_k, S_i \rangle = 0$ for $1 \leq k \leq i-1$.
  [Initially, S_1 is any arbitrary non-zero vector in the space $\{0,1\}^N$.]
  Compute a minimum weight cycle C_i such that $\langle C_i, S_i \rangle = 1$.
end

Lemma 3.1. *For each i, $1 \leq i \leq N$ there is at least one cycle C that satisfies $\langle C, S_i \rangle = 1$.*

Proof. Observe that each S_i is non-zero and thus it has to contain at least one edge $e = (u, v) \in E \setminus T$. The cycle C_e formed by the unique path on T from u to v and edge e has intersection of size exactly 1 with S_i. Thus, there is always at least one cycle that satisfies $\langle C_e, S_i \rangle = 1$. □

Lemma 3.2 shows that the set $\{C_1, \ldots, C_N\}$ returned by Algorithm 3.2 is a basis.

Lemma 3.2. *For each i, $1 \leq i \leq N$ cycle C_i is linearly independent of cycles $C_1, \ldots, C_{i-1}$.*

Proof. Any vector v in the span of $\{C_1, \ldots, C_{i-1}\}$ satisfies $\langle v, S_i \rangle = 0$ since $\langle C_j, S_i \rangle = 0$ for all $1 \leq j \leq i-1$. But C_i satisfies $\langle C_i, S_i \rangle = 1$. Thus, C_i cannot lie in the span of $\{C_1, \ldots, C_{i-1}\}$ or we get a contradiction. □

We next show that it is also a minimum cycle basis.

Theorem 3.3 (de Pina [20])**.** *The set of cycles $\{C_1, \ldots, C_N\}$ computed by Algorithm 3.2 is a minimum cycle basis.*

Proof. Suppose not. Then, there exists some $0 \leq i < N$ such that there is a minimum cycle basis $\mathbb{B}$ that contains $\{C_1, \ldots, C_i\}$ but there is no minimum cycle basis that contains $\{C_1, \ldots, C_i, C_{i+1}\}$. Since the cycles in $\mathbb{B}$ form a spanning set, there exist cycles $B_1, \ldots, B_k$ in $\mathbb{B}$ such that

$$C_{i+1} = B_1 + B_2 + \cdots + B_k \ . \tag{3.1}$$

Since $\langle C_{i+1}, S_{i+1} \rangle = 1$, there exists some B_j in the above sum such that $\langle B_j, S_{i+1} \rangle = 1$. But C_{i+1} is a minimum weight cycle such that $\langle C_{i+1}, S_{i+1} \rangle = 1$. So the weight of $C_{i+1} \leq$ the weight of B_j.

Let $\mathbb{B}' = \mathbb{B} \cup \{C_{i+1}\} \setminus \{B_j\}$. Since B_j is equal to the sum of C_{i+1} and $\{B_1, \ldots, B_k\} \setminus \{B_j\}$ (refer Equation (3.1)), $\mathbb{B}'$ is also a basis. Moreover, $\mathbb{B}'$ has weight at most the weight of $\mathbb{B}$ which is a minimum cycle basis. So $\mathbb{B}'$ is also a minimum cycle basis.

We have $\{C_1, C_2, \ldots, C_{i+1}\} \subseteq \mathbb{B}'$ because by assumption $\{C_1, \ldots, C_i\} \subseteq \mathbb{B}$ and the cycle B_j that was omitted from $\mathbb{B}$ cannot be equal to any one of $C_1, \ldots, C_i$ since $\langle B_j, S_{i+1} \rangle = 1$ whereas $\langle C_l, S_{i+1} \rangle = 0$ for all $l \leq i$. The existence of the minimum cycle basis $\mathbb{B}'$ that contains the cycles $\{C_1, \ldots, C_{i+1}\}$ contradicts our assumption that there is no minimum cycle basis containing $\{C_1, \ldots, C_i, C_{i+1}\}$. Hence, the cycles $\{C_1, C_2, \ldots, C_N\}$ are indeed a minimum cycle basis. □

There are two main subroutines in Algorithm 3.2:

(a) computing a non-zero vector S_i in the subspace orthogonal to the cycles $\{C_1, \ldots, C_{i-1}\}$,

(b) computing a minimum weight cycle C_i such that $\langle C_i, S_i \rangle = 1$.

Depending on how we implement these two steps we can obtain algorithms with different running time complexities. Especially for the computation of S_i, depending on the amount of information that we are reusing from phases $1, \ldots, i-1$, we can obtain better and better running times.

3.3 Computing the Cycles

Given S_i we can compute a minimum weight cycle C_i such that $\langle C_i, S_i \rangle = 1$ by reducing it to n shortest paths computations in an appropriate graph G_i. The following construction is well known [6, 43]. The *signed graph* G_i is defined from $G = (V, E)$ and $S_i \subseteq E$ in the following manner.

G_i has two copies of each vertex $v \in V$. Call them v^+ and v^-.
for every edge $e = (v, u) \in E$ **do**
  if $e \notin S_i$ **then**
    Add edges (v^+, u^+) and (v^-, u^-) to the edge set of G_i.
    {Assign their weights to be the same as e.}
  else
    Add edges (v^+, u^-) and (v^-, u^+) to the edge set of G_i.
    {Assign their weights to be the same as e.}
  end if
end for

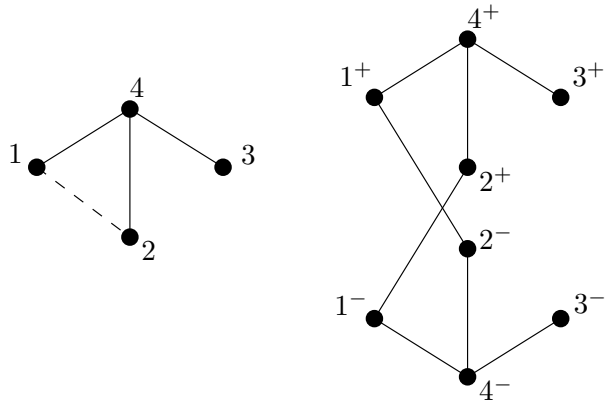


Figure 3.2: An example of the graph G_i, where $S_i = \{(1,2)\}$. Since the edge $(1,2)$ belongs to S_i we have the edges $(1^-, 2^+)$ and $(1^+, 2^-)$ going across the $-$ and $+$ levels. The edges not in S_i, i.e., $(1,4)$, $(2,4)$, and $(3,4)$ have copies inside the $+$ level and the $-$ level.

G_i can be visualized as 2 levels of G, the $+$ level and the $-$ level. Within each level we have edges of $E \setminus S_i$. Between the levels we have the edges of S_i. See Figure 3.2 for an example of G_i.

Given any v^+ to v^- path in G_i, we can correspond to it a cycle in G by identifying the vertices and edges in G_i with their corresponding vertices and edges in G. Because we identify both v^+ and v^- with v, any v^+ to v^- path p in G_i corresponds to a cycle C in G. More formally, take the incidence vector of the path p (over the edges of G_i) and obtain an incidence vector over the edges of G by identifying $(v^*, u^\dagger)$ with (v, u) where $*$ and $\dagger$ are $+$ or $-$. Suppose the path p contains more than one copy of the same edge (it could for example contain both (v^+, u^-) and (v^-, u^+) for some (v, u)). Then, add the number of occurrences of that edge modulo 2 to obtain an incidence vector over the edges of G. As an example, consider Figure 3.2. The path $\langle 4^+, 1^+, 2^-, 4^- \rangle$ corresponds to the cycle $\langle 4, 1, 2, 4 \rangle$ in G.

Lemma 3.4. *The path $p = \min_{v \in V}$ shortest (v^+, v^-) path in G_i corresponds to a minimum weight cycle C in G that has odd intersection with S_i.*

Proof. Since the endpoints of the path p are v^+ and v^-, p has to contain an odd number of edges of S_i. This is because only edges of S_i provide a change of sign and p goes from a $+$ vertex to a $-$ vertex. We might have deleted some edges of S_i while forming C since those edges occurred with a multiplicity of 2. But this means that we always delete an even number of edges from S_i. Hence, C has an odd number of edges of S_i present in

it. Also, the weight of $C \leq$ the weight (or length) of p since edges have non-negative weights.

We should now prove that C is a minimum weight cycle among such cycles. Let C' be any other cycle in G with an odd number of edges of S_i in it. If C' is not a simple cycle, then C' is a union of simple cycles (with disjoint edges) and at least one of those simple cycles C_0 should have an odd number of edges of S_i present in it. Note also that the weight of $C_0 \leq$ the weight of C'.

Let u be a vertex in C_0. We will identify C_0 with a path in G_i by traversing C_0 starting at the vertex u and identifying it with u^+. If we traverse an edge e of S_i, then we identify the vertices incident on e with opposite signs. If we traverse an edge outside S_i, then we identify the vertices incident on e with the same sign. Since C_0 is a cycle, we come back to the vertex u. Also, C_0 has an odd number of edges of S_i present in it. So the sign of the final vertex is of the opposite sign to the sign of the starting vertex. Hence, C_0 translates to a u^+ to u^- path p' in G_i and the weight of $p' =$ the weight of C_0.

But p was the minimum weight path among all shortest (v^+, v^-) paths in G_i for all $v \in V$. Hence, the weight of $p \leq$ the weight of p'. So we finally get that the weight of $C \leq$ the weight of $p \leq$ the weight of $p' \leq$ the weight of C'. This proves that C is a minimum weight cycle that has odd intersection with S_i. □

The computation of the path p can be performed:

(a) by computing n shortest (v^+, v^-) paths, one for each vertex $v \in V$, each by Dijkstra's algorithm in G_i and taking their minimum, or

(b) by one invocation of an all pairs shortest paths algorithm in G_i.

This computation takes $O(n(m + n \log n))$. Note that depending on the relation between m and n, the algorithm can choose which shortest paths algorithm to use. For example, in the case when the edge weights are integers or the unweighted case it is better to use faster all pairs shortest paths algorithms than run Dijkstra's algorithm n times.

In the general case of weighted undirected graphs, since we have to compute in total N such cycles $C_1, C_2, \ldots, C_N$, we spend $O(mn(m + n \log n))$ time, since $N = m - n + 1$.

3.4 Computing the Witnesses

We now consider the problem of computing the subsets S_i, for $1 \leq i \leq N$. We want S_i to be a non-zero vector in the subspace orthogonal to $\{C_1, \ldots, C_{i-1}\}$. The trivial way would be to solve one linear system in each of the N iterations. This would cost $O(m^\omega)$ in each iteration and thus a total of $O(m^{\omega+1})$.

One way to improve upon the trivial way is to maintain a whole basis of the subspace. Any vector in that basis will then be a non-zero vector in the subspace. The intuition behind such an approach is that in each iteration we have only one new cycle, and thus computing a basis of the orthogonal subspace should be relatively easy since we know the basis from the previous iteration. This is the approach chosen by de Pina [20] in Algorithm 3.1. We now cast this approach to our algebraic framework and prove its correctness.

Initially, $S_j = \{e_j\}$ for all j, $1 \leq j \leq N$. This corresponds to the standard basis of the space $\{0,1\}^N$. At the beginning of phase i, we have $\{S_i, S_{i+1}, \ldots, S_N\}$ which is a basis of the space $\mathcal{C}^\perp$ orthogonal to the space $\mathcal{C}$ spanned by $\{C_1, \ldots, C_{i-1}\}$. We use S_i to compute C_i and update vectors $\{S_{i+1}, \ldots, S_N\}$ to a basis $\{S'_{i+1}, \ldots, S'_N\}$ of the subspace of $\mathcal{C}^\perp$ that is orthogonal to C_i. The update step of phase i is as follows:
For $i+1 \leq j \leq N$, let

$$S'_j = \begin{cases} S_j & \text{if } \langle C_i, S_j \rangle = 0 \ , \\ S_j + S_i & \text{if } \langle C_i, S_j \rangle = 1 \ . \end{cases}$$

The following lemma proves that this step does indeed what we claim.

Lemma 3.5. *The set $\{S'_{i+1}, \ldots, S'_N\}$ forms a basis of the subspace orthogonal to $\{C_1, \ldots, C_i\}$.*

Proof. We will first show that $S'_{i+1}, \ldots, S'_N$ belong to the subspace orthogonal to $C_1, \ldots, C_i$. We know that $S_i, S_{i+1}, \ldots, S_N$ form a basis of the subspace orthogonal to $C_1, \ldots, C_{i-1}$. Since each S'_j, $i+1 \leq j \leq N$ is a linear combination of S_j and S_i, it follows that S'_j is orthogonal to $C_1, \ldots, C_{i-1}$. If an S_j is already orthogonal to C_i, then we leave it as it is, i.e., $S'_j = S_j$. Otherwise $\langle C_i, S_j \rangle = 1$ and we update S_j as $S'_j = S_j + S_i$. Since both $\langle C_i, S_j \rangle$ and $\langle C_i, S_i \rangle$ are equal to 1, it follows that each S'_j is now orthogonal to C_i also. Hence, $S'_{i+1}, \ldots, S'_N$ belong to the subspace orthogonal to $C_1, \ldots, C_i$.

Now we will show that $S'_{i+1}, \ldots, S'_N$ are linearly independent. Suppose there is a linear dependence among them. Substitute S'_j's in terms of S_j's and S_i in the linear dependence relation. S_i is the only vector that might

occur more than once in this relation. So either S_i occurs an even number of times and gets cancelled and we get a linear dependence among $S_{i+1}, \ldots, S_N$ or S_i occurs an odd number of times, in which case we get a linear dependence among $S_i, S_{i+1}, \ldots, S_N$. Either case contradicts the linear independence of $S_i, S_{i+1}, \ldots, S_N$. We conclude that $S'_{i+1}, \ldots, S'_N$ are linearly independent. □

This completes the description of the algebraic framework (see Algorithm 3.2) and one of its possible implementations (see Algorithm 3.1). Let us now bound the running time of Algorithm 3.1. During the update step of the i-th iteration, the cost of updating each S_j, $j > i$ is $O(N)$ and hence it is $O(N(N-i))$ for updating $S_{i+1}, \ldots, S_N$. There are N iterations in total, thus, the total cost of maintaining this basis is $O(N^3)$ which is $O(m^3)$.

The total running time of the Algorithm 3.1 is $O(m^3 + mn^2 \log n)$ by summing up the costs of computing the cycles and witnesses. For dense graphs the bottleneck of Algorithm 3.1 is the $O(m^3)$ term, meaning that the choice of the shortest paths method is not crucial for the worst case running time.

3.5 A New Algorithm

In this section we are going to realize Algorithm 3.2 in such a manner such that the cost of updating the witnesses is $O(m^\omega)$. This also implies that different shortest paths routines will give us different running time bounds.

Recall our approach to compute the vectors S_i. We maintained a basis of $\mathcal{C}^\perp$ in each iteration and that required $O(m^2)$ in each iteration. Note that we need just one vector from the subspace orthogonal to $C_1, \ldots, C_i$. But the algorithm maintains $N - i$ such vectors: $S_{i+1}, \ldots, S_N$. This is the limiting factor in the running time of the algorithm. In order to improve the running time of Algorithm 3.2, we relax the invariant that $S_{i+1}, \ldots, S_N$ form a basis of the subspace orthogonal to $C_1, \ldots, C_i$. Since we need just one vector in this subspace, we can afford to relax this invariant and maintain the correctness of the algorithm.

In Algorithm 3.2, as realized in Section 3.4, in the i-th iteration we update $S_{i+1}, \ldots, S_N$. The idea now is to update only those S_j's where j is close to i and postpone the update of the later S_j's. During the postponed update, many S_j's can be updated simultaneously. This simultaneous update is implemented as a matrix multiplication step and the use of a fast algorithm for matrix multiplication causes the speedup.

Algorithm 3.3: A faster MCB algorithm.

Initialize the cycle basis with the empty set and initialize $S_j = \{e_j\}$ for $1 \leq j \leq N$.

Call the procedure *extend_cb*$(\{\}, \{S_1, \ldots, S_N\}, N)$.

A call to *extend_cb*$(\{C_1, \ldots, C_i\}, \{S_{i+1}, \ldots, S_{i+k}\}, k)$ extends the cycle basis by k cycles. Let $\mathcal{C}$ denote the current partial cycle basis which is $\{C_1, \ldots, C_i\}$.

Procedure *extend_cb*$(\mathcal{C}, \{S_{i+1}, \ldots, S_{i+k}\}, k)$**:**

if $k = 1$ **then**
  compute a minimum weight cycle C_{i+1} such that $\langle C_{i+1}, S_{i+1} \rangle = 1$.
else
  call *extend_cb*$(\mathcal{C}, \{S_{i+1}, \ldots, S_{i+\lfloor k/2 \rfloor}\}, \lfloor k/2 \rfloor)$ to extend the current cycle basis by $\lfloor k/2 \rfloor$ elements. That is, we compute the cycles $C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}$ in a recursive manner.

  During the above recursive call, $S_{i+1}, \ldots, S_{i+\lfloor k/2 \rfloor}$ get updated. Denote their final versions (at the end of this step) as $S'_{i+1}, \ldots, S'_{i+\lfloor k/2 \rfloor}$.

  call *update*$(\{S'_{i+1}, \ldots, S'_{i+\lfloor k/2 \rfloor}\}, \{S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}\})$ to update $S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}$. Let $T_{i+\lfloor k/2 \rfloor+1}, \ldots, T_{i+k}$ be the output returned by *update*.

  call *extend_cb*$(\mathcal{C} \cup \{C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}\}, \{T_{i+\lfloor k/2 \rfloor+1}, \ldots, T_{i+k}\}, \lceil k/2 \rceil)$ to extend the current cycle basis by $\lceil k/2 \rceil$ cycles. That is, the cycles $C_{i+\lfloor k/2 \rfloor+1}, \ldots, C_{i+k}$ will be computed recursively.
end

Our main procedure is called *extend_cb* and works in a recursive manner. Algorithm 3.3 contains a succinct description.

Procedure *extend_cb*$(\{C_1, \ldots, C_i\}, \{S_{i+1}, \ldots, S_{i+k}\}, k)$ computes k new cycles $C_{i+1}, \ldots, C_{i+k}$ of the MCB using the subsets $S_{i+1}, \ldots, S_{i+k}$. We maintain the invariant that these subsets are all orthogonal to $C_1, \ldots, C_i$. It first computes $C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}$ using $S_{i+1}, \ldots, S_{i+\lfloor k/2 \rfloor}$. At this point, the remaining subsets $S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}$ need not be orthogonal to the new cycles $C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}$. Our algorithm then updates $S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}$ so that they are orthogonal to cycles $C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}$ and they continue to be orthogonal to $C_1, \ldots, C_i$. Then, it computes the remaining cycles $C_{i+\lfloor k/2 \rfloor+1}, \ldots, C_{i+k}$.

Let us see a small example as to how this works. Suppose $N = 4$. We initialize the subsets S_i, $i = 1, \ldots, 4$ and call *extend_cb*, which then calls itself with only S_1 and S_2 and then only with S_1 and so computes C_1. Then, it updates S_2 so that $\langle C_1, S_2 \rangle = 0$ and computes C_2. Then, it simultaneously updates S_3 and S_4 which were still at their initial values so that the updated S_3 and S_4 (which we call T_3 and T_4) are both orthogonal to C_1 and C_2. Next it computes C_3 using T_3 and updates T_4 to be orthogonal to C_3. T_4 was already orthogonal to C_1 and C_2 and the update step maintains this. Finally, it computes C_4.

Observe that whenever we compute C_{i+1} using S_{i+1}, we have the property that S_{i+1} is orthogonal to $C_1, \ldots, C_i$. The difference is the function *update* which allows us to update many S_j's simultaneously to be orthogonal to many C_i's. As mentioned earlier, this simultaneous update enables us to use the fast matrix multiplication algorithm which is crucial to the speedup. We next describe the update step in detail.

The function *update*

When we call function $update(\{S'_{i+1}, \ldots, S'_{i+\lfloor k/2 \rfloor}\}, \{S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}\})$, the sets $S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}$ need not all be orthogonal to the space spanned by $\mathcal{C} \cup \{C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}\}$. We already know that $S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}$ are all orthogonal to $\mathcal{C}$ and now we need to ensure that the updated sets $S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}$ (call them $T_{i+\lfloor k/2 \rfloor+1}, \ldots, T_{i+k}$) are all orthogonal to $\mathcal{C} \cup \{C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}\}$. We now want to update the sets $S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}$, i.e., we want to determine $T_{i+\lfloor k/2 \rfloor+1}, \ldots, T_{i+k}$ such that for each j in the range for $i + \lfloor k/2 \rfloor + 1 \leq j \leq i + k$ we have

(i) T_j is orthogonal to $C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}$, and

(ii) T_j remains orthogonal to $C_1, \ldots, C_i$.

So, we define T_j (for each $i + \lfloor k/2 \rfloor + 1 \leq j \leq i + k$) as follows:

$$T_j = S_j + \text{a linear combination of } S'_{i+1}, \ldots, S'_{i+\lfloor k/2 \rfloor} \ . \tag{3.2}$$

This makes sure that T_j is orthogonal to the cycles $C_1, \ldots, C_i$ because S_j and all of $S'_{i+1}, \ldots, S'_{i+\lfloor k/2 \rfloor}$ are orthogonal to $C_1, \ldots, C_i$. Hence, T_j which is a linear combination of them will also be orthogonal to $C_1, \ldots, C_i$. The coefficients of the linear combination will be chosen such that T_j will be orthogonal to $C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}$. Rewriting Equation (3.2) we get

$$T_j = S_j + a_{j1} S'_{i+1} + a_{j2} S'_{i+2} + \cdots + a_{j\lfloor k/2 \rfloor} S'_{i+\lfloor k/2 \rfloor} \ .$$

We will determine the coefficients $a_{j1}, \ldots, a_{j\lfloor k/2 \rfloor}$ for all $i + \lfloor k/2 \rfloor + 1 \leq j \leq i + k$ simultaneously. Writing all these equations in matrix form, we have

$$\begin{pmatrix} T_{i+\lfloor k/2 \rfloor+1} \\ \vdots \\ \vdots \\ T_{i+k} \end{pmatrix} = (A \; I) \cdot \begin{pmatrix} S'_{i+1} \\ \vdots \\ S'_{i+\lfloor k/2 \rfloor} \\ S_{i+\lfloor k/2 \rfloor+1} \\ \vdots \\ S_{i+k} \end{pmatrix} \tag{3.3}$$

where A is the $\lceil k/2 \rceil \times \lfloor k/2 \rfloor$ matrix whose ℓ-th row has the unknowns $a_{j1}, \ldots, a_{j\lfloor k/2 \rfloor}$, where $j = i + \lfloor k/2 \rfloor + \ell$. Here T_j represents a row with the coefficients of T_j as its row elements.

Let

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} S'_{i+1} \\ \vdots \\ S'_{i+\lfloor k/2 \rfloor} \\ S_{i+\lfloor k/2 \rfloor+1} \\ \vdots \\ S_{i+k} \end{pmatrix} \cdot \left(C_{i+1}^T \ldots C_{i+\lfloor k/2 \rfloor}^T \right) \tag{3.4}$$

where

$$X = \begin{pmatrix} S'_{i+1} \\ \vdots \\ S'_{i+\lfloor k/2 \rfloor} \end{pmatrix} \cdot \left(C_{i+1}^T \ldots C_{i+\lfloor k/2 \rfloor}^T \right) \tag{3.5}$$

and

$$Y = \begin{pmatrix} S_{i+\lfloor k/2 \rfloor+1} \\ \vdots \\ S_{i+k} \end{pmatrix} \cdot \left(C_{i+1}^T \ldots C_{i+\lfloor k/2 \rfloor}^T \right) . \tag{3.6}$$

Let us multiply both sides of Equation (3.3) with an $N \times \lfloor k/2 \rfloor$ matrix whose columns are the cycles $C_{i+1}, \ldots, C_{i+\lfloor k/2 \rfloor}$. Using Equation (3.4) we get

$$\begin{pmatrix} T_{i+\lfloor k/2 \rfloor+1} \\ \vdots \\ \vdots \\ T_{i+k} \end{pmatrix} \cdot \left(C_{i+1}^T \ldots C_{i+\lfloor k/2 \rfloor}^T \right) = (A \; I) \cdot \begin{pmatrix} X \\ Y \end{pmatrix} . \tag{3.7}$$

The left hand side of Equation (3.7) is the 0 matrix since each of the vectors $T_{i+\lfloor k/2\rfloor+1}, \ldots, T_{i+k}$ has to be orthogonal to each of $C_{i+1}, \ldots, C_{i+\lfloor k/2\rfloor}$.

The following lemma shows that X is invertible, or in other words that the coefficients we are looking for do indeed exist.

Lemma 3.6. *The matrix X in Equation (3.5) is invertible.*

Proof. The matrix

$$X = \begin{pmatrix} \langle C_{i+1}, S'_{i+1}\rangle & \ldots & \langle C_{i+\lfloor k/2\rfloor}, S'_{i+1}\rangle \\ \langle C_{i+1}, S'_{i+2}\rangle & \ldots & \langle C_{i+\lfloor k/2\rfloor}, S'_{i+2}\rangle \\ \vdots & \vdots & \vdots \\ \langle C_{i+1}, S'_{i+\lfloor k/2\rfloor}\rangle & \ldots & \langle C_{i+\lfloor k/2\rfloor}, S'_{i+\lfloor k/2\rfloor}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & * & * & \ldots & * \\ 0 & 1 & * & \ldots & * \\ 0 & 0 & 1 & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

is an upper triangular matrix with 1's on the diagonal, since each S'_j is the final version of the subset S_j using which C_j is computed, which means that $\langle C_j, S'_j\rangle = 1$ and $\langle C_\ell, S'_j\rangle = 0$ for all $\ell < j$. Hence, X is invertible. □

We are thus given an invertible $\lfloor k/2\rfloor \times \lfloor k/2\rfloor$ matrix X and a $\lceil k/2\rceil \times \lfloor k/2\rfloor$ matrix Y and we want to find a $\lceil k/2\rceil \times \lfloor k/2\rfloor$ matrix A such that:

$$(A\ I) \cdot \begin{pmatrix} X \\ Y \end{pmatrix} = 0\ .$$

Here 0 stands for the $\lceil k/2\rceil \times \lfloor k/2\rfloor$ zero-matrix and I stands for the $\lceil k/2\rceil \times \lceil k/2\rceil$ identity matrix.

We need $AX + Y = 0$ or $A = -YX^{-1} = YX^{-1}$ since we are in the field $\mathbb{F}_2$. We can determine A in time $O(k^\omega)$ using fast matrix multiplication and inverse algorithms since the matrix X is invertible (Lemma 3.6). Hence, we can compute all the coefficients $a_{j1}, \ldots, a_{j\lfloor k/2\rfloor}$ for all $i+\lfloor k/2\rfloor+1 \le j \le i+k$ simultaneously using matrix multiplication and matrix inversion algorithms.

By the implementation of the function *update*, Lemma 3.7 follows.

Lemma 3.7. *In the case $k = 1$, i.e., whenever procedure extend_cb is called like extend_cb($\{C_1, \ldots, C_i\}, S_{i+1}, 1$), the vector S_{i+1} is orthogonal to the cycles $\{C_1, \ldots, C_i\}$. Moreover, S_{i+1} always contains the edge e_{i+1}.*

Corollary 3.8. *At the end of Algorithm 3.3 the $N \times N$ matrix whose i-th row is S_i for $1 \leq i \leq N$ is lower triangular with 1 in its diagonal.*

Proof. By the implementation of update we know that the final version of the vector S_i contains edge e_i. Moreover, the final version of the vector S_i is a linear combination of S_i and S_j for $j < i$. It is easy to show by induction that the final version of S_i does not contain any of the edges $e_{i+1}, \ldots, e_N$. □

Hence, just before we compute C_{i+1} we always have a non-zero vector S_{i+1} orthogonal to $\{C_1, \ldots, C_i\}$. Moreover, C_{i+1} is a minimum weight cycle such that $\langle C_{i+1}, S_{i+1} \rangle = 1$. The correctness of Algorithm 3.3 follows from Theorem 3.3.

Theorem 3.9. *The set of cycles $\{C_1, \ldots, C_N\}$ computed by Algorithm 3.3 is a minimum cycle basis.*

3.5.1 Running Time

Let us analyze the running time of Algorithm 3.3. The recurrence of the algorithm is as follows:

$$T(k) = \begin{cases} \text{cost of computing a minimum weight cycle } C_i \\ \text{such that } \langle C_i, S_i \rangle = 1 & \text{if } k = 1, \\ 2T(k/2) + \text{cost of update} & \text{if } k > 1. \end{cases}$$

The computation of matrices X and Y takes time $O(mk^{\omega-1})$ using the fast matrix multiplication algorithm. To compute X (respectively Y) we are multiplying $\lfloor k/2 \rfloor \times N$ by $N \times \lfloor k/2 \rfloor$ (respectively $\lceil k/2 \rceil \times N$ by $N \times \lfloor k/2 \rfloor$) matrices. We split the matrices into $2N/k$ square blocks and use fast matrix multiplication to multiply the blocks. Thus, multiplication takes time $O((2N/k)(k/2)^\omega) = O(mk^{\omega-1})$. We can also invert X in $O(k^\omega)$ time and we also multiply Y and X^{-1} using fast matrix multiplication in order to get the matrix A. Finally, we use the fast matrix multiplication algorithm again, to multiply the matrix $(A\ I)$ with the matrix whose rows are $S'_{i+1}, \ldots, S'_{i+\lfloor k/2 \rfloor}, S_{i+\lfloor k/2 \rfloor+1}, \ldots, S_{i+k}$. This way we get the updated subsets $T_{i+\lfloor k/2 \rfloor+1}, \ldots, T_{i+k}$ (refer to Equation (3.3)). As before this multiplication can be performed in time $O(mk^{\omega-1})$.

Using the algorithm described in Section 3.3 to compute a shortest cycle C_i that has odd intersection with S_i, the recurrence turns into

$$T(k) = \begin{cases} O(mn + n^2 \log n) & \text{if } k = 1, \\ 2T(k/2) + O(k^{\omega-1}m) & \text{if } k > 1. \end{cases}$$

This solves to $T(k) = O(k(mn+n^2 \log n)+k^{\omega-1}m)$. Thus, $T(m) = O(m^\omega + m^2n + mn^2 \log n)$. Since $m^\omega < m^2n$, this reduces to $T(m) = O(m^2n + mn^2 \log n)$.

For $m > n \log n$, this is $T(m) = O(m^2n)$. For $m \leq n \log n$, this is $T(m) = O(mn^2 \log n)$. We have shown the following theorem.

Theorem 3.10. *A minimum cycle basis in an undirected weighted graph can be computed in time $O(m^2n + mn^2 \log n)$.*

Our algorithm has a running time of $O(m^\omega + m \cdot n(m + n \log n))$, where the $n(m + n \log n)$ term is the cost to compute all pairs shortest paths. We can also formulate a more general theorem.

Theorem 3.11. *A minimum cycle basis in an undirected weighted graph with m edges and n vertices can be computed in time $O(m^\omega + m \cdot \text{APSP}(m, n))$ where $\text{APSP}(m, n)$ denotes the time to compute all pairs shortest paths.*

When the edges of G have integer weights, we can compute all pairs shortest paths in time $O(mn)$ [76, 77], that is, we can bound $T(1)$ by $O(mn)$. When the graph is unweighted or the edge weights are small integers, we can compute all pairs shortest paths in time $\tilde{O}(n^\omega)$ [72, 36]. When such graphs are reasonably dense, say $m \geq n^{1+1/(\omega-1)} \operatorname{poly}(\log n)$, then the $O(m^\omega)$ term dominates the running time of our algorithm. We conclude with the following corollary.

Corollary 3.12. *A minimum cycle basis in an undirected graph with integer edge weights can be computed in time $O(m^2n)$. For unweighted graphs which satisfy $m \geq n^{1+1/(\omega-1)} \operatorname{poly}(\log n)$, for some fixed polynomial, we have an $O(m^\omega)$ algorithm to compute a minimum cycle basis.*

3.6 Computing a Certificate of Optimality

In this section we address the problem of constructing a certificate to verify a claim that a given set of cycles $\mathcal{C} = \{C_1, \ldots, C_N\}$ forms an MCB. A certificate is an "easy to verify" witness of the optimality of our answer.

The sets S_i for $1 \leq i \leq N$ in our algorithm, from which we calculate the cycles $\mathcal{C} = \{C_1, \ldots, C_N\}$ of the minimum cycle basis, are a certificate of the optimality of $\mathcal{C}$. The verification algorithm would consist of verifying

that the cycles in $\mathcal{C}$ are linearly independent and that each C_i is a minimum weight cycle such that $\langle C_i, S_i \rangle = 1$.

Asymptotically the verification algorithm and Algorithm 3.3 have the same running time. However, the verification algorithm is conceptually much simpler and the constants involved much smaller. Thus, given a set of cycles $\{C_1, \ldots, C_N\}$ we would like to compute its certificate. Algorithm 3.4 computes witnesses $S_1, \ldots, S_N$ given $C_1, \ldots, C_N$.

Algorithm 3.4: Given a set of cycles, compute its certificate.

Compute a spanning tree T. Let $\{e_1, \ldots, e_N\}$ be the edges of $E \setminus T$.
Form the 0-1 $N \times N$ matrix $\mathcal{C} = \left(C_1^T \ldots C_N^T\right)$, where the i-th column of $\mathcal{C}$ is the incidence vector of C_i over $\{e_1, \ldots, e_N\}$.
Compute $\mathcal{C}^{-1}$.
if *the matrix inversion algorithm returns an error* **then**
    $\mathcal{C}$ is singular. Thus, $C_1, \ldots, C_N$ are linearly dependent. Return an error since they cannot form a cycle basis.
else
    Return the rows of $\mathcal{C}^{-1}$ as our witnesses or certificate.
end

The rows of $\mathcal{C}^{-1}$ form our witnesses $S_1, S_2, \ldots, S_N$. The property that we want from $S_1, \ldots, S_N$ is that for each $1 \leq i \leq N$, $\langle C_i, S_i \rangle = 1$. Since $\mathcal{C}^{-1}\mathcal{C}$ is the identity matrix, this property is obeyed by the rows of $\mathcal{C}^{-1}$.

Suppose each C_i is a minimum weight cycle such that $\langle C_i, S_i \rangle = 1$. Then, by Lemma 4.1 we have that $\sum_{i=1}^{N} |C_i| \leq$ weight of an MCB. Since $\{C_1, \ldots, C_N\}$ are linearly independent (by the existence of $\mathcal{C}^{-1}$), it means that $\{C_1, \ldots, C_N\}$ forms a minimum cycle basis.

On the other hand, if for some i, C_i is not a minimum weight cycle such that $\langle C_i, S_i \rangle = 1$, then by replacing C_i with a minimum weight cycle that has odd intersection with S_i (as in the proof of Theorem 3.3) we get a cycle basis with smaller weight.

Hence, the cycles $\{C_1, \ldots, C_N\}$ form an MCB if and only if each C_i is a minimum weight cycle such that $\langle C_i, S_i \rangle = 1$. Since the inverse of an $N \times N$ matrix can be computed in $O(N^\omega)$ time, we have the following theorem.

Theorem 3.13. *Given a set of cycles $\mathcal{C} = \{C_1, \ldots, C_N\}$ we can construct a certificate $\{S_1, \ldots, S_N\}$ in $O(m^\omega)$ time.*

3.7 Concluding Remarks

In this chapter we have presented an algorithm to solve the minimum cycle basis problem in undirected weighted graphs in $O(m^2n + mn^2 \log n)$ time. The algorithm is a result of presenting an algebraic framework based on the work of de Pina [20] and then implementing the linear independence step faster using fast matrix multiplication. Using specialized shortest paths for the shortest path step of the algebraic framework, we also provide algorithms with improved running times for special cases like integer edge weights or when the graph is unweighted.

The main remaining open problem is whether the running time of algorithms based on this algebraic framework can be improved and reach the $O(m^\omega)$ upper bound even for sparse graphs.

Chapter 4

Approximate Minimum Cycle Basis

Summary

Most algorithms that use cycle bases build and solve some linear system based on such a basis. The use of a minimum cycle basis is not required but provides the sparsest such linear system and thus results in faster running times. However, the best running time required to compute a minimum cycle basis is still not really practical.

In this chapter we compute approximate minimum cycle bases. Such bases are still considered sparse and moreover can be computed much faster. We present an α-approximate algorithm for any $\alpha > 1$. Although faster than the exact approach this algorithm does not drop below the $\Theta(m^\omega)$ bound. On the other hand, for any integer $k \geq 1$ we present two constant factor $2k-1$ approximate algorithms which for sufficiently dense graphs are $o(m^\omega)$. The first algorithm which is faster for sparser graphs has an expected running time of $O(kmn^{1+2/k} + mn^{(1+1/k)(\omega-1)})$ while the second has deterministic running time $O(n^{3+2/k})$.

Our techniques are based on spanner constructions. For graphs which admit better spanners we provide even faster algorithms. These techniques extend also to the directed minimum cycle basis problem. We give very fast approximate algorithms for this version as well.

4.1 Introduction

The most obvious application of the minimum cycle basis problem is to construct sparse systems when solving problems in electrical networks [74, 20]. The recent work of Berger et al. [8] is directly motivated by this application. Furthermore, the problem has many applications in many diverse areas of science. One such application is for example structural engineering [14], while chemistry and biochemistry [39] is another. Recently, Tewari et al. [75] used a minimum cycle basis to do surface reconstruction from a point cloud sampled from a genus one smooth manifold.

In most cases the use of minimum cycle bases is done as a preprocessing

step. The use of an MCB ensures sparseness and therefore results into faster running times since the amount of work is dependent on the cycle basis chosen. Because MCB algorithms are mostly used as preprocessing, their running time should not dominate the whole algorithm's running time. Unfortunately this is not always the case. Algorithm 3.3 has a running time of $O(m^2n + mn^2 \log n)$. In some special cases of graphs this becomes $\Theta(m^\omega)$ but not lower.

This chapter presents several approximation algorithms for computing approximate minimum cycle bases in undirected graphs. The running times are significant improvements over the exact approach. In the case of constant factor approximation we present algorithms which are $o(m^\omega)$ for sufficiently dense graphs.

4.2 An α-approximate[†] Algorithm

The bottleneck in the running time of our exact minimum cycle basis algorithm is the computation of a minimum weight cycle C_i such that $\langle C_i, S_i \rangle = 1$. Suppose we relax our constraint that our cycle basis should have minimum weight and ask for a cycle basis whose weight is at most α times the weight of an MCB.

Algorithm 4.1: An α-approximation MCB algorithm.

for $i = 1$ *to* N **do**

Let S_i be any arbitrary non-zero vector in the subspace orthogonal to $\{D_1, D_2, \ldots, D_{i-1}\}$, i.e., S_i satisfies: $\langle D_k, S_i \rangle = 0$ for $1 \leq k \leq i-1$.

Compute a cycle D_i such that $\langle D_i, S_i \rangle = 1$ and the weight of $D_i \leq \alpha \cdot$ the weight of a minimum weight cycle that has odd intersection with S_i.

end

For any parameter $\alpha > 1$, we present an approximation algorithm which computes a cycle basis whose weight is at most α times the weight of a minimum cycle basis. To the best of our knowledge, this is the first time that an approximation algorithm for the MCB problem is being given.

[†]Formally, an *α-approximation algorithm* for a minimization problem Π is a polynomial time algorithm, that for any instance I of problem Π always computes a feasible solution to I, whose value is at most a factor α of the value of the optimum solution to I. We call α the *approximation ratio*, or an *approximation* (or *performance*) *guarantee*.

This algorithm is obtained by relaxing the base step ($k = 1$) in procedure *extend_cb* of Algorithm 3.3. In the original algorithm we computed a minimum weight cycle C_{i+1} such that $\langle C_{i+1}, S_{i+1}\rangle = 1$. Here, we relax it to compute a cycle D_{i+1} such that $\langle D_{i+1}, S_{i+1}\rangle = 1$ and the weight of D_{i+1} is at most α times the weight of a minimum weight cycle that has odd intersection with S_{i+1} (call such a cycle, α-stretch). The method of updating the subsets S_i is identical to the way the update step is done in Algorithm 3.3.

Algorithm 4.1 describes the general framework of our approximation algorithm in order to compute a set of cycles $\{D_1, \ldots, D_N\}$.

The linear independence of the D_i's follows from the existence of the S_i's. That is, $\langle D_i, S_i\rangle = 1$ while $\langle D_k, S_i\rangle = 0$ for all $1 \leq k \leq i-1$ shows that D_i is linearly independent of $D_1, \ldots, D_{i-1}$. Similarly, note that the subsets $S_1, \ldots, S_N$ are linearly independent since each S_i is independent of $S_{i+1}, \ldots, S_N$ because $\langle D_i, S_i\rangle = 1$ whereas $\langle D_i, S_j\rangle = 0$ for each $j > i$.

It remains to prove the correctness of Algorithm 4.1. Let $|C|$ denote the weight of cycle C. We need to show that $\sum_{i=1}^{N} |D_i| \leq \alpha \cdot$ weight of MCB. Let A_i be a shortest cycle that has odd intersection with S_i. The set $\{A_1, \ldots, A_N\}$ need not be linearly independent since the subsets S_i's were not updated according to the A_i's. The following lemma was originally shown in [20] in order to give an equivalent characterization of the MCB problem as a maximization problem. We present a simple proof of the lemma here.

Lemma 4.1 (de Pina [20])**.** *Let $S_1, \ldots, S_N$ be linearly independent vectors in $\{0,1\}^N$ and let A_i be the shortest cycle in G such that $\langle A_i, S_i\rangle = 1$. Then, $\sum_{i=1}^{N} |A_i| \leq w(\text{MCB})$.*

Proof. We will look at the A_i's in sorted order, i.e., let π be a permutation on $[N]$ such that $|A_{\pi(1)}| \leq |A_{\pi(2)}| \leq \ldots \leq |A_{\pi(N)}|$. Let $C_1, \ldots, C_N$ be the cycles of an MCB and let $|C_1| \leq |C_2| \leq \ldots \leq |C_N|$. We will show that for each i, $|A_{\pi(i)}| \leq |C_i|$. That will prove the lemma.

We will first show that $\langle C_k, S_{\pi(\ell)}\rangle = 1$ for some k and ℓ with $1 \leq k \leq i \leq \ell \leq N$. Otherwise, the $N - i + 1$ linearly independent vectors $S_{\pi(i)}, S_{\pi(i+1)}, \ldots, S_{\pi(N)}$ belong to the subspace orthogonal to $C_1, \ldots, C_i$; however, this subspace has dimension only $N - i$.

Thus, $|A_{\pi(\ell)}| \leq |C_k|$ since $A_{\pi(\ell)}$ is a shortest cycle s.t. $\langle A_{\pi(\ell)}, S_{\pi(\ell)}\rangle = 1$. But by the sorted order, $|A_{\pi(i)}| \leq |A_{\pi(\ell)}|$ and $|C_k| \leq |C_i|$. This implies that $|A_{\pi(i)}| \leq |C_i|$. □

Theorem 4.2. *The linearly independent cycles $\{D_1, \ldots, D_N\}$ computed in Algorithm 4.1 have weight at most α times the weight of a minimum cycle basis.*

Proof. By construction for each $1 \leq i \leq N$ we know that $|D_i| \leq \alpha \cdot |A_i|$. Using Lemma 4.1 we get that

$$\sum_{i=1}^{N} |D_i| \leq \alpha \sum_{i=1}^{N} |A_i| \leq \alpha \cdot \text{weight of MCB} . \qquad \square$$

4.2.1 Running Time

Since all the steps of Algorithm 4.1 except the base step corresponding to computing a cycle are identical to Algorithm 3.3, we have the following recurrence for Algorithm 4.1:

$$T(k) = \begin{cases} \text{cost of computing an } \alpha\text{-stretch cycle } D_i \text{ such} \\ \text{that } \langle D_i, S_i \rangle = 1 & \text{if } k = 1, \\ 2T(k/2) + O(k^{\omega-1}m) & \text{if } k > 1. \end{cases}$$

So the running time of Algorithm 4.1 depends on which parameter α is used in the algorithm. We will compute an α-stretch cycle D_i that is odd in S_i by using the same method as in Section 3.3. But instead of a shortest (v^+, v^-) path in G_i, here we compute an α-stretch (v^+, v^-) path. We next show that the minimum of such paths corresponds to an α-stretch cycle in G that has odd intersection with S_i. The proof is very similar to the proof of Lemma 3.4.

Lemma 4.3. *The path $p = \min_{v \in V}$ α-stretch (v^+, v^-) path in G_i corresponds to an α-stretch cycle C in G that has odd intersection with S_i.*

Proof. Since the endpoints of the path p are v^+ and v^-, p has to contain an odd number of edges of S_i. This is because only edges of S_i provide a change of sign and p goes from a $+$ vertex to a $-$ vertex. We might have deleted some edges of S_i while forming C since those edges occurred with a multiplicity of 2. But this means that we always delete an even number of edges from S_i. Hence, C has an odd number of edges of S_i present in it. Also, the weight of $C \leq$ the weight (or length) of p since edges have non-negative weights.

We should now prove that C is an α-stretch cycle among such cycles. Let C' be any other cycle in G with an odd number of edges of S_i in it. If C' is not a simple cycle, then C' is a union of simple cycles (with disjoint edges) and at least one of those simple cycles C_0 should have an odd number of edges of S_i present in it. Note also that the weight of $C_0 \leq$ the weight of C'.

Let u be a vertex in C_0. We will identify C_0 with a path in G_i by traversing C_0 starting at the vertex u and identifying it with u^+. If we traverse an edge e of S_i, then we identify the vertices incident on e with opposite signs. If we traverse an edge outside S_i, then we identify the vertices incident on e with the same sign. Since C_0 is a cycle, we come back to the vertex u. Also, C_0 has an odd number of edges of S_i present in it. So the sign of the final vertex is of the opposite sign to the sign of the starting vertex. Hence, C_0 translates to a u^+ to u^- path p' in G_i and the weight of p' = the weight of C_0.

But p is the minimum weight path among α-stretch (v^+, v^-) paths in G_i for all $v \in V$. Hence, the weight of $p \leq \alpha$ times the weight of p'. So we finally get that the weight of $C \leq$ the weight of $p \leq \alpha$ times the weight of $p' \leq \alpha$ times the weight of C'. This proves that C is an α-stretch cycle that has odd intersection with S_i. □

We formulate our theorem in its general form.

Theorem 4.4. *An α-approximate minimum cycle basis in an undirected weighted graph with m edges and n vertices can be computed in time $O(m^\omega + m \cdot \text{APASP}(\alpha, m, n))$ where $\text{APASP}(\alpha, m, n)$ denotes the time to compute all pairs α-approximate shortest paths.*

When $\alpha = 2$, we use the result in [16] to compute 2-stretch paths which would result in 2-stretch cycles. Then, Algorithm 4.1 runs in time $\tilde{O}(m^{3/2}n^{3/2}) + O(m^\omega)$. For reasonably dense graphs (for example, $m \geq n^{(1.5+\delta)/(\omega-1.5)}$ for a constant $\delta > 0$), this is an $O(m^\omega)$ algorithm.

At this point we should make the following remark. In each phase we need to find the minimum among n approximate paths but we need to actually construct at most one of these paths. We would like to avoid the complication of traversing all n paths in order to select the minimum, as this could potentially dominate the running time. This could be the case for example if the approximate shortest paths algorithm returns non-simple paths. For $\alpha = 2$ the result by Cohen and Zwick computes distances between all pairs of vertices which are 2-stretch. In this case we first find the pair (v^+, v^-) which achieves the minimum that we are looking for and then simply perform an exact shortest path computation only between this pair. This results in time $\tilde{O}(m^{1/2}n^{3/2}) + O(m + n \log n)$ which is $\tilde{O}(m^{1/2}n^{3/2})$ in each phase. Summing over all the N phases we get the following.

Corollary 4.5. *A cycle basis which is a 2-approximate minimum cycle basis in an undirected weighted graph can be computed in $\tilde{O}(m^{3/2}n^{3/2}) + O(m^\omega)$ time.*

For $(1+\epsilon)$-approximation we use the all pairs $(1+\epsilon)$-stretch paths algorithm [81]. Then, we have an $\tilde{O}(\frac{mn^\omega}{\epsilon}\log\frac{W}{\epsilon})+O(m^\omega)$ algorithm to compute a cycle basis which is at most $1+\epsilon$ times the weight of an MCB, where W is the largest edge weight in the graph. If $m \geq n^{1+1/(\omega-1)}\operatorname{poly}(\log n)$ and all edge weights are polynomial in n, then Algorithm 4.1 is an $O(\frac{m^\omega}{\epsilon}\log\frac{1}{\epsilon})$ algorithm.

As before in each phase we use the approximate shortest paths algorithm by Zwick in order to select the (v^+, v^-) pair which achieves the minimum approximate path. Then, we only construct this path by doing an exact shortest path computation. Thus, outputting the cycle basis takes $O(m + n\log n)$ in each phase or $O(m^2 + mn\log n)$ in total.

Corollary 4.6. *A $(1+\epsilon)$-approximate minimum cycle basis in an undirected weighted graph, for any $\epsilon > 0$, can be computed in time $\tilde{O}(\frac{mn^\omega}{\epsilon}\log\frac{W}{\epsilon}) + O(m^\omega)$ where W is the largest edge weight in the graph. The time to output the approximate minimum cycle basis is at most $O(m^2 + mn\log n)$.*

The previous approximation algorithm can approximate arbitrarily close to the optimum but its running time is not lower than $\Theta(m^\omega)$. Next we present algorithms which for sufficiently dense graphs are $o(m^\omega)$. The idea is simple. We divide the computation in two phases. In the first phase we compute, very fast, a large number of cycles. In the second phase we compute the remaining cycles using the techniques that we have developed so far.

4.3 Most of the Cycles

The lower bound in Lemma 4.1 is quite powerful. It also provides us with the following corollary which triggers the development of faster approximation algorithms.

Lemma 4.7. *Consider a weighted undirected graph $G(V, E)$ and let T be an arbitrary spanning tree of G. Let C_i for $1 \leq i \leq N$ be the cycle formed by edge $e_i = (u_i, v_i) \in E \setminus T$ and the shortest path in $G \setminus \{e_i\}$ between u_i and v_i. Then, $\sum_{i=1}^{N} |C_i| \leq w(\text{MCB})$.*

Proof. Set $R_i = \{e_i\}$ for $1 \leq i \leq N$ in Lemma 4.1. □

We call the above set of cycles the *short cycles multi-set*. The fact that the minimum cycle basis is lower bounded by the short cycles multi-set leads to the observation that a large subset of an approximate minimum cycle basis can be computed efficiently if we are given a sparse t-spanner

of G. Let $\mathrm{SP}_G(u,v)$ denote the shortest path in G from u to v and let $w(\mathrm{SP}_G(u,v))$ denote its weight. When it is clear from the context we simply write $\mathrm{SP}(u,v)$.

Definition 4.1 (multiplicative spanner)**.** A multiplicative t-spanner of a graph G is a subgraph $G'(V,E')$, $E' \subseteq E$ s.t. for any $v,u \in V$ we have $w(\mathrm{SP}_G(u,v)) \leq w(\mathrm{SP}_{G'}(u,v)) \leq t \cdot w(\mathrm{SP}_G(u,v)$.

Let $G'(V,E')$ be such a t-spanner of G. We will fix details like its size or the parameter t later on. For now let us present the main idea. For each edge $e=(u,v) \in E \setminus E'$ we will compute the cycle formed by e and $\mathrm{SP}_{G'}(u,v)$. Since edges in $E \setminus E'$ do not belong to E' each of these cycles contains an edge that it not contained in any other cycle. This ensures that the computed cycles are linearly independent. For edge $e=(u,v) \in E \setminus E'$ this cycle has weight at most t times the weight of the shortest cycle in G containing e. Denote by λ the cardinality of $E \setminus E'$. Thus, for $E \setminus E' = \{e_1, \ldots, e_\lambda\}$ we can compute a set of $\lambda \leq N$ cycles $\{C_1, \ldots, C_\lambda\}$ which are linearly independent and each cycle C_i has weight at most t times the length of the shortest cycle in G containing edge e_i. By Lemma 4.7 the weight of the short cycles multi-set is a lower bound for the MCB and thus the computed set of cycles has weight at most t times the weight of the MCB. Note that this set might not have the right number of elements and thus is not necessarily a basis.

The computation of these cycles can be performed by either λ single source shortest paths computations in the spanner G' or by one all-pairs shortest paths computation. We will ensure that this spanner is sparse and thus this computation is relatively cheap.

Running Time

We need to compute a sparse spanner of our input graph G. As pointed out by Althöfer et al. [1] every weighted undirected graph on n vertices has a $(2k-1)$-spanner with $O(n^{1+1/k})$ where $k \geq 1$ is an integer. Such a spanner can be constructed using an algorithm similar to Kruskal's algorithm (see Cormen et al. [18]) for constructing minimum spanning trees. In order to build the spanner consider all edges of the graph in non-decreasing order of weight, adding each edge to the spanner if its endpoints are not already connected, in the spanner, by a path using at most $2k-1$ edges. At any stage, the spanner is a $(2k-1)$-spanner of the edges already considered, and its unweighted girth (number of edges of the smallest simple cycle) is at least $2k+1$, so it has only $O(n^{1+1/k})$ edges (see [1, 10]). The above procedure can be implemented in $O(mn^{1+1/k})$.

Algorithm 4.2: The first $(2k-1)$-approximation algorithm.

input : Graph $G(V, E)$ and integer $k \geq 1$.
output: A $(2k-1)$-approximate MCB.

Construct a $(2k-1)$-spanner G' with $O(n^{1+1/k})$ edges. Let $e_1, \ldots, e_\lambda$ be the edges of $G \setminus G'$.

For each $1 \leq i \leq \lambda$ construct C_i as $e_i = (u_i, v_i)$ and the shortest path in G' from u_i to v_i.

Find linearly independent $S_{\lambda+1}, \ldots, S_N$ in the subspace orthogonal to cycles $C_1, \ldots, C_\lambda$.

Call Algorithm 3.3 as $extend\_cb(\{C_1, \ldots, C_\lambda\}, \{S_{\lambda+1}, \ldots, S_N\}, N-\lambda)$ to compute $(2k-1)$-approximate cycles $C_{\lambda+1}, \ldots, C_N$, i.e., the basis of the recursion uses $(2k-1)$-approximate distance oracles.

Return $\{C_1, \ldots, C_\lambda\} \cup \{C_{\lambda+1}, \ldots, C_N\}$.

In the above spanner we are going to perform λ shortest paths computations, one for each edge of G that is not in the spanner. Using Dijkstra's algorithm we need $O(\lambda \cdot (n^{1+1/k} + n \log n))$ and since $\lambda \leq m$ we can compute both the spanner and the λ linearly independent cycles in time $O(mn^{1+1/k})$. At this point we should mention that there are faster algorithms to construct similar spanners, see for example [78, 71]. Nevertheless, at this phase the above solution is satisfactory since it is not the dominating factor of the running time.

4.4 The Remaining Cycles

In Section 4.3 we computed most of the cycles of an approximate minimum cycle basis. We are left with computing the remaining cycles. Note that the number of the remaining cycles is exactly the dimension of the cycle space of the spanner G'. By ensuring that the spanner G' is sparse the remaining cycles will be close to linear. We present three algorithms which are based on two different approaches. Each of them is faster for different graph densities. The main difference between the two approaches is whether we are going to use the edges $e_1, \ldots, e_\lambda$ in the remaining cycles.

4.4.1 1st Approach

Our first approach computes the remaining cycles by performing approximate cycles computations in G.

We are going to use Algorithm 3.3 in order to compute the remaining cycles. As a base case we will use approximate cycles computations. Let μ denote the remaining cycles to compute. First of all we know that $\lambda + \mu = N$. Consider now the first λ cycles $C_1, \ldots, C_\lambda$ that we have already computed in Section 4.3. We need to find sets $S_{\lambda+1}, \ldots, S_N$ which are a basis of the orthogonal subspace of $C_1, \ldots, C_\lambda$. Assume for now that we have such a basis. We can execute Algorithm 3.3. The algorithm will first recursively compute cycles $C_{\lambda+1}, \ldots, C_{\lambda+\lfloor(N-\lambda)/2\rfloor}$ by using and modifying $S_{\lambda+1}, \ldots, S_{\lambda+\lfloor(N-\lambda)/2\rfloor}$. Then, in one bulk step will update $S_{\lambda+\lfloor(N-\lambda)/2\rfloor+1}, \ldots, S_N$ to be orthogonal to $C_{\lambda+1}, \ldots, C_{\lambda+\lfloor(N-\lambda)/2\rfloor}$. Note that $S_{\lambda+\lfloor(N-\lambda)/2\rfloor+1}, \ldots, S_N$ were already orthogonal to $C_1, \ldots, C_\lambda$ and the update step maintains this. Finally by using and modifying sets $S_{\lambda+\lfloor(N-\lambda)/2\rfloor+1}, \ldots, S_N$ the algorithm will recursively compute the last cycles $C_{\lambda+\lfloor(N-\lambda)/2\rfloor+1}, \ldots, C_N$.

This will cost $T(\mu) = \mu \cdot T(1) + O(m\mu^{\omega-1})$. The fact that μ is $o(n^2)$ results in the improvement of the running time. The basis of the recursion is to compute a t-approximate shortest cycle C_i which has an odd intersection with the corresponding set S_i for $\lambda + 1 \leq i \leq N$.

Null Space Basis

In order to continue our computation of the remaining cycles we look for a basis $S_{\lambda+1}, \ldots, S_N$ of the orthogonal subspace of the cycles $C_1, \ldots, C_\lambda$. Consider the cycles $C_1, \ldots, C_\lambda$ which we view (as always) as vectors in the space $\{0,1\}^N$. Let us write them in an array, one row per cycle,

$$\begin{pmatrix} C_1 \\ \vdots \\ C_\lambda \end{pmatrix} = \begin{pmatrix} I_\lambda \; B \end{pmatrix}$$

where I_λ here is the $\lambda \times \lambda$ identity matrix and B is a $\lambda \times \mu$ matrix. The matrix has this form since each of the edges e_i for $1 \leq i \leq \lambda$ belongs only to the cycle C_i. The rows of the above matrix are linearly independent.

We would like to find a basis of the null space of this matrix. We set

$$\begin{pmatrix} S_{\lambda+1} \; \ldots \; S_N \end{pmatrix} = \begin{pmatrix} B \\ I_\mu \end{pmatrix},$$

where I_μ stands for the $\mu \times \mu$ identity matrix. The product of the two matrices is $B + B = 0$ where 0 stands for the $\lambda \times \mu$ zero-matrix , i.e., the S's are orthogonal to the C's. Moreover, the S's are linearly independent.

By this definition, the $S_{\lambda+1}, \ldots, S_N$ have the following property: they contain some edges from $\{e_1, \ldots, e_\lambda\}$ and $e_{\lambda+i} \in S_{\lambda+i}$ for $1 \leq i \leq \mu$. This property is important for the existence of the cycles with odd intersection with the witnesses while running Algorithm 3.3.

The running time required to compute this null space basis is the time required to output the already known matrix B. By using some sparse representation of the vectors we need at most $O(\lambda \cdot \mu)$. In the general case $\lambda \leq m$ and $\mu = N - \lambda \in O(n^{1+1/k})$. Thus, this step needs $O(mn^{1+1/k})$ time.

Remaining Cycles

The final step is to execute Algorithm 3.3 in order to compute the remaining cycles. The input to this algorithm is cycles $C_1, \ldots, C_\lambda$ and sets $S_{\lambda+1}, \ldots, S_N$. As mentioned earlier this requires time $T(\mu) = \mu \cdot T(1) + O(m\mu^{\omega-1})$. Here $T(1)$ is the time to perform t-approximate single source shortest paths between at most n pairs of vertices in the signed graph G_i.

The shortest paths computation is going to be performed $\mu = N - \lambda$ times. Therefore, we require a faster algorithm than constructing a spanner. We are going to use an *approximate distance oracle*. Thorup and Zwick [78] constructed a structure which answers $(2k-1)$-approximate shortest paths in time $O(k)$. This structure requires space $O(kn^{1+1/k})$ and can be constructed in expected time $O(kmn^{1/k})$. In the case of unweighted graphs Baswana and Sen [7] showed that a $(2k-1)$-approximate distance oracle can be computed in expected $O(\min(n^2, kmn^{1/k}))$ time. In the unweighted case the same can also be done deterministically [70].

We use such a construction. The cost of computing cycles $C_{\lambda+1}, \ldots, C_N$ in the last part is $\mu \cdot T(1) + O(m\mu^{\omega-1})$. We bound $T(1)$ by the cost of computing an approximate distance oracle of G_i (the graph introduced in Section 3.3) which is $O(kmn^{1/k})$ expected time and the cost of performing n queries in the approximate distance oracle. Each query costs $O(k)$ and thus a total of $O(nk)$.

Note that although we perform n queries to this oracle we only need to build one path. Given a query (u, v) the approximate distance oracle of Thorup and Zwick finds a tree, constructed during preprocessing, which contains both u and v and the path in this tree between u and v is a $2k-1$ stretch path. Thus, the returned path is simple and it has at most $O(n)$ edges. We can construct it explicitly in amortized constant time per edge or by some preprocessing in constant time per edge. Thus, forming the actual cycle can be done in $O(n)$ time.

Therefore, the total cost is $O(\mu(kn + kmn^{1/k}) + m\mu^{\omega-1})$. Since $\mu \in$

$O(n^{1+1/k})$ we get a bound of $O(kn^{2+1/k}+kmn^{1+2/k}+mn^{(1+1/k)(\omega-1)})$ which is $O(kmn^{1+2/k}+mn^{(1+1/k)(\omega-1)})$ assuming that $m \geq n$.

Approximation Guarantee

Let us now prove that the computed set of cycles is a t-approximation of the MCB. Let $S_{\lambda+1}, \ldots, S_N$ be the witnesses at the end of the algorithm. At this point we should emphasize that the sets $S_{\lambda+1}, \ldots, S_N$ change during the execution of Algorithm 3.3. Each computed cycle $C_{\lambda+1}, \ldots, C_N$ is a t-approximation of the shortest cycle in G with an odd intersection with the corresponding witness. Moreover, when the algorithm computes a cycle C_i for $\lambda+1 \leq i \leq N$ the set S_i has acquired its final value and is orthogonal with all cycles $C_1, \ldots, C_{i-1}$. Let also $S_i = \{e_i\}$ for $1 \leq i \leq \lambda$. Then, each C_i for $1 \leq i \leq \lambda$ is also a t-approximation of the shortest cycle in G with an odd intersection with S_i.

Lemma 4.8. *The sets $S_1, \ldots, S_N$ are linearly independent.*

Proof. Consider S_i. We know that $\langle C_i, S_i \rangle = 1$ for all $1 \leq i \leq N$. We claim that $\langle C_i, S_j \rangle = 0$ for all $i+1 \leq j \leq N$. In this case each S_i is independent of the $S_{i+1}, \ldots, S_N$ and the lemma follows.

Assume a partition of the set $\{S_{i+1}, \ldots, S_N\}$ into two sets $\{S_{i+1}, \ldots, S_\lambda\}$ and $\{S_{\lambda+1}, \ldots, S_N\}$. The claim follows for the first set from the definition of the sets $S_{i+1}, \ldots, S_\lambda$. For $1 \leq i \leq \lambda$ the cycle C_i contains edge e_i and a path in the spanner. The spanner has edges e_j only for $j > \lambda$ and sets $S_{i+1}, \ldots, S_\lambda$ by definition contain only edges from $\{e_{i+1}, \ldots, e_\lambda\}$. For the second set the claim follows by the invariants of Algorithm 3.3 since we initialize $S_{\lambda+1}, \ldots, S_N$ to be orthogonal to C_i and the update step maintains this. □

It is now straightforward to show that the computed set of cycles is a t-approximation of a minimum cycle basis. Combine Lemma 4.8, Lemma 4.1 and the fact that for each $1 \leq i \leq N$ we compute a cycle C_i which t-approximates the shortest cycle with an odd intersection with S_i. We set $t = 2k-1$ and obtain the following theorem.

Theorem 4.9. *A $(2k-1)$-approximate minimum cycle basis, for any integer $k \geq 1$, in an undirected weighted graph can be computed in expected time $O(kmn^{1+2/k}+mn^{(1+1/k)(\omega-1)})$.*

We also obtain the following corollary.

Corollary 4.10. *An $O(\log n)$-approximate minimum cycle basis in an undirected weighted graph can be computed in expected time $O(mn^{\omega-1}+mn\log n)$.*

Algorithm 4.3: The second $(2k-1)$-approximation algorithm.

input : Graph $G(V, E)$ and integer $k \geq 1$.
output: A $(2k-1)$-approximate MCB.

Construct a $(2k-1)$-spanner G' with $O(n^{1+1/k})$ edges. Let $e_1, \ldots, e_\lambda$ be the edges of $G \setminus G'$.

For each $1 \leq i \leq \lambda$ construct C_i as $e_i = (u_i, v_i)$ and the shortest path in G' from u_i to v_i.

Call Algorithm 3.3 with exact shortest paths to find an MCB $C_{\lambda+1}, \ldots, C_N$ of G'.

Return $\{C_1, \ldots, C_\lambda\} \cup \{C_{\lambda+1}, \ldots, C_N\}$.

4.4.2 2nd Approach

Our second algorithm just computes a minimum cycle basis of the t-spanner G'. The dimension of the cycle space of G' is $\mu = N - \lambda$ and thus we have the right number of cycles. Let $C_1, \ldots, C_\lambda$ be the cycles computed in Section 4.3 and $C_{\lambda+1}, \ldots, C_N$ be the MCB of G'. Cycles $\{C_1, \ldots, C_\lambda\} \cup \{C_{\lambda+1}, \ldots, C_N\}$ are by definition linearly independent and we are also going to prove that they form a t-approximation of the MCB of G.

For $1 \leq i \leq \lambda$, let p_i denote the shortest path in G' between u_i and v_i where $e_i = (u_i, v_i)$. Recall that for $1 \leq i \leq \lambda$ we have $C_i = e_i + p_i$. We assume that the edges in E are indexed s.t. $e_1, \ldots, e_\lambda$ are the edges in $E \setminus E'$, edges $e_{\lambda+1}, \ldots, e_N$ belong to E' and edges $e_{N+1}, \ldots, e_m$ are the edges of a fixed but arbitrary spanning tree T of G' and thus also of G.

In order to show that cycles $C_1, \ldots, C_N$ are a t-approximation of the MCB we again define appropriate linearly independent vectors $S_1, \ldots, S_N \in \{0,1\}^N$ and use Lemma 4.1. Consider Algorithm 3.3 executing with input the t-spanner G'. Except from cycles $C_{\lambda+1}, \ldots, C_N$ it returns also vectors (edge sets in G') $R_{\lambda+1}, \ldots, R_N \in \{0,1\}^{N-\lambda}$ s.t. for each $\lambda + 1 \leq i \leq N$ and $i < j \leq N$ we have $\langle C_i, R_j \rangle = 0$ and moreover C_i is the shortest cycle in G' s.t. $\langle C_i, R_i \rangle = 1$ for $\lambda + 1 \leq i \leq N$. Also, the $N - \lambda \times N - \lambda$ matrix whose j-th row is R_j is lower triangular with 1 in its diagonal (see Corollary 3.8). This implies that the R_j's are linearly independent. Given any vector $S \in \{0,1\}^N$ let $S^\dagger$ be the projection of S onto its last $N - \lambda$ coordinates. In other words if S is an edge set of G let $S^\dagger$ be the edge set restricted only to the edges of G'.

We define vectors $S_j \in \{0,1\}^N$ for $1 \leq j \leq N$ as follows. For $1 \leq j \leq \lambda$ let S_j be the vector with 0 everywhere except the j coordinate which is 1.

For $\lambda + 1 \leq j \leq N$ let

$$S_j = (-\langle C_1^\dagger, R_j\rangle, \ldots, -\langle C_\lambda^\dagger, R_j\rangle, R_{j1}, R_{j2}, \ldots, R_{j(N-\lambda)}) \ .$$

Note that the above defined sets S_j for $1 \leq j \leq N$ are linearly independent. This follows by construction since the $N \times N$ matrix whose j-th row is S_j is lower triangular with 1's in its diagonal. The above definition of S_j's is motivated by the property that for each $1 \leq i \leq \lambda$, we have $\langle C_i, S_j\rangle = -\langle C_i^\dagger, R_j\rangle + \langle C_i^\dagger, R_j\rangle = 0$, since the cycle C_i has 0 in all first λ coordinates except the i-th coordinate which is 1.

We are now in the position to show the following which together with Lemma 4.1 implies the correctness of our approach.

Lemma 4.11. *Consider the above defined sets S_j for $1 \leq j \leq N$ and let D_j be the shortest cycle in G s.t. $\langle D_j, S_j\rangle = 1$. Cycle C_j returned by Algorithm 4.3 has weight at most t times the weight of D_j.*

Proof. This is trivial for $1 \leq j \leq \lambda$ since D_j is the shortest cycle in G which uses edge e_j and $C_j = e_j + p_j$. Consider now D_j for $\lambda + 1 \leq j \leq N$. If D_j uses any edge e_i for $1 \leq i \leq \lambda$ we replace it with the corresponding shortest path in the spanner. This is as saying consider the cycle $D_j + C_i$ instead of D_j. Let

$$D'_j = D_j + \sum_{1 \leq i \leq \lambda} (e_i \in D_j) C_i$$

where $(e_i \in D_j)$ is 1 if $e_i \in D_j$ and 0 if $e_i \notin D_j$. Then,

$$\langle D'_j, S_j\rangle = \langle D_j, S_j\rangle + \sum_{1 \leq i \leq \lambda} (e_i \in D_j)\langle C_i, S_j\rangle \ .$$

But for $1 \leq i \leq \lambda$ the cycle C_i has 0 in all first λ coordinates except the i-th one which is 1. Thus, by the definition of S_j we have $\langle C_i, S_j\rangle = -\langle C_i^\dagger, R_j\rangle + \langle C_i^\dagger, R_j\rangle = 0$. We conclude that $\langle D'_j, S_j\rangle = \langle D_j, S_j\rangle = 1$. But D'_j by definition has 0 in the first λ coordinates and $S_j^\dagger = R_j$, which in turn implies that $\langle D_j'^\dagger, R_j\rangle = 1$.

C_j is the shortest cycle in G' s.t. $\langle C_j, R_j\rangle = 1$. Thus, C_j has weight at most the weight of $D_j'^\dagger$ (which is the same cycle as D'_j) and by construction D'_j has weight at most t times the weight of D_j. □

Altogether, we have shown that the cost of our approximate basis is at most t times the cost of an optimal basis. As a t-spanner we again use a $(2k-1)$-spanner. The best time bound in order to compute an MCB is

Algorithm 4.4: A $(2k-1)(2q-1)$-approximation algorithm.

input : Graph $G(V, E)$ and integers $k, q \geq 1$.
output: A $(2q-1)(2k-1)$-approximate MCB.

Construct a $(2k-1)$-spanner G' with $\tilde{O}(kn^{1+1/k})$ edges. Let $e_1, \ldots, e_\lambda$ be the edges of $G \setminus G'$.

For each $1 \leq i \leq \lambda$ construct C_i as $e_i = (u_i, v_i)$ and a $(2k-1)$-stretch path from u_i to v_i obtained by the spanner G'.

Call Algorithm 4.2 to compute a $(2q-1)$-approximate MCB of G'. Denote these cycles as $C_{\lambda+1}, \ldots, C_N$

Return $\{C_1, \ldots, C_\lambda\} \cup \{C_{\lambda+1}, \ldots, C_N\}$.

$O(m^2n + mn^2 \log n)$ and since a $(2k-1)$-spanner has at most $O(n^{1+1/k})$ edges the total running time becomes $O(n^{3+2/k})$.

Theorem 4.12. *A $(2k-1)$-approximate minimum cycle basis, for any integer $k \geq 1$, in a weighted undirected graph can be computed in time $O(n^{3+2/k})$.*

Since any graph has an $O(\log n)$-spanner of linear size we also obtain the following corollary.

Corollary 4.13. *An $O(\log n)$-approximate* MCB *in a weighted undirected graph can be computed in time $O(n^3 \log n)$.*

Algorithm 4.2 is faster for sparser graphs and Algorithm 4.3 for denser graphs. More precisely, Algorithm 4.3 is faster if $m > n^{4-\omega+\frac{3-\omega}{k}}$ which with the current upper bound on ω is $m > n^{1.624+\frac{0.624}{k}}$. Both algorithms are faster than $O(m^\omega)$, Algorithm 4.3 for $m > n^{(3+2/k)/\omega} = n^{1.26+\frac{0.84}{k}}$ and Algorithm 4.2 for $m > \max(n^{1+1/k}, \sqrt[\omega-1]{kn^{1+2/k}})$.

4.4.3 More Approximation

Consider now Algorithm 4.3. After computing the first λ cycles we use an exact MCB algorithm to compute the MCB of the t-spanner G'. We can instead use our approximate Algorithm 4.2 which is fast for sparse graphs.

We begin by computing a $(2k-1)$-spanner G' of G and the first λ cycles. Then, we call Algorithm 4.2 to compute a $(2q-1)$-approximate MCB of G'. The result will be a $(2k-1)(2q-1)$-approximate MCB of G. This follows by the fact that our first λ cycles combined with the MCB of G' form a $(2k-1)$-approximate MCB.

Note that in this case it is not sufficient to use the $O(mn^{1+1/k})$ approach of Althöfer et al. [1] in order to compute the initial $(2k-1)$-spanner G'.

Thorup and Zwick [78] have shown how to construct a $(2k-1)$-spanner G' with size $\tilde{O}(kn^{1+1/k})$ in expected time $\tilde{O}(kmn^{1/k})$. Such a spanner is a collection of shortest path trees $T(w)$ for $w \in V$ and given two vertices $u, v \in V$ there is a tree in this collection that contains a path between u and v that is of stretch at most $2k-1$. Furthermore, the corresponding tree can be obtained in $O(k)$ time. We can also preprocess these trees in order to be able to output the path in amortized or worst case constant time per edge.

Consider running Algorithm 4.2 in the subgraph G'. The algorithm will first compute a $(2q-1)$-spanner G'' of the $(2k-1)$-spanner G' and some of the cycles. Then, it will compute the remaining cycles by doing $(2q-1)$-approximate shortest cycles computations in G'. We therefore get an algorithm with expected running time $\tilde{O}(qk \cdot n^{1+1/k} n^{1+2/q} + kn^{1+1/k} n^{(1+1/q)(\omega-1)})$. Note that this upper bound does not include the time to output the initial cycles using the $(2k-1)$-spanner.

Theorem 4.14. *A $(2k-1)(2q-1)$-approximate* MCB *of an undirected graph with non-negative edge weights, for any two integers $k, q \geq 1$, can be computed in expected time $\tilde{O}(qk \cdot n^{2+1/k+1/q} \cdot (n^{1/q} + n^{(1+1/q)(\omega-2)}))$ plus the time to output the* MCB*.*

Corollary 4.15. *An $O(\log^2 n)$ approximate* MCB *of an undirected weighted graph can be computed in expected time $\tilde{O}(n^\omega)$ plus the time to output the* MCB*.*

4.5 Planar and Euclidean Graphs

Some types of graphs are much easier to handle than others w.r.t the minimum cycle basis. Consider a planar graph G. In linear time we can find an embedding of G in the plane. Then, the approximate cycle basis algorithm just returns the set of bounded faces of G. It is easy to see that all these cycles are linearly independent and due to Euler's formula they are also the right number, $N = m - n + 1 = f - 1$, where f is the number of faces of G.

For the approximation factor let $E^* \subseteq E$ be the set of edges of G which belong to at least one cycle. Since each edge is incident with at most two faces, this cycle basis has weight at most

$$2 \cdot \sum_{e \in E^*} w(e) \leq 2 \cdot w(\text{MCB}) \ ,$$

since a minimum cycle basis has to use each edge that belongs to at least one cycle.

The embedding can be found in linear time [50]. The bounded faces can also be enumerated in linear time and the size of the output is also linear.

Theorem 4.16. *A 2-approximate minimum cycle basis of a planar graph can be computed in linear time using linear space.*

Let P be a set of points in the Euclidean plane. The Euclidean graph $G(P)$ on P has vertices corresponding to points in P. The graph is complete and the weight of an edge is the Euclidean distance between the points corresponding to the vertices. In order to find an approximate MCB of $G(P)$ we use the result that the Delaunay triangulation of the points P is a ≈ 2.42 [59] spanner of $G(P)$. Let DT denote the Delaunay triangulation of P. DT is a planar graph. The approximate MCB algorithm returns the following set of cycles: (a) For each edge $e = (u, v) \in G(P) \setminus \mathrm{DT}$ the cycle C_e formed by edge e and the shortest path in DT from u to v, and (b) a minimum cycle basis of DT.

Theorem 4.17. *A 2.42 approximation of the MCB in a complete Euclidean graph in the plane can be constructed in $O(n^3)$ time.*

Proof. The proof of the approximation ratio follows exactly the proof in Section 4.4.2. For the running time we note that computing the Delaunay triangulation requires $O(n \log n)$ time. The shortest paths computations on the planar spanner require $O(\lambda n)$ which is $O(n^3)$. Computing an MCB of a planar graph can be done in $O(n^2 \log n)$ by the algorithm of Hartvigsen and Mardon [48]. □

In the case we want to handle higher (but fixed) dimensions we can use the fact that the well-separated pair decomposition [13] (s-WSPD) for separation factor $s = 4(t+1)/(t-1)$ is a t-spanner of $G(P)$ for $t > 1$. The algorithm begins by computing an s-WSPD decomposition. The time to compute an s-WSPD is $O(n \log n + s^d n)$ where d is the dimension. Since the size of the WSPD is $O(s^d n)$ we know that $\lambda \leq m$ and $\mu = N - \lambda \in O(s^d n)$. Thus, the time to compute the first λ cycles is $O(m(s^d n + n \log n))$.

For the remaining cycles we use the approach in Algorithm 4.3. Computing exactly the MCB of the spanner we get a running time of $O(s^d n^3 \log n)$.

Theorem 4.18. *A t-approximate MCB for $t > 1$ of a complete Euclidean graph can be computed in time $O(s^d n^3 \log n)$ where $s = 4(t+1)/(t-1)$ and d the fixed dimension.*

4.6 Directed Graphs

Our techniques can also be applied to the minimum cycle basis problem in directed graphs. A cycle in a directed graph is a cycle in the underlying undirected graph with edges traversed in both directions. A $\{-1, 0, 1\}$ edge incidence vector is associated with each cycle: edges traversed by the cycle in the right direction get 1 and edges traversed in the opposite direction get -1. The main difference here is that the cycle space is generated over $\mathbb{Q}$ by the cycle vectors. Note that the weight of a cycle is simply the sum of the weight of its edges independent of the orientation of these edges.

The algorithms for finding an MCB in directed graphs are based on the techniques used in [20] and the ones developed in this thesis. Several extra ideas are required in order to compensate for the extra cost of arithmetic that arises when changing the base field from $\mathbb{F}_2$ to $\mathbb{Q}$. Lemma 4.1 can be generalized, see for example [56]. Algorithm 4.3 can also be directly generalized. For the spanner computation we view our directed graph G as undirected and we compute a $(2k-1)$-spanner G'. We then give to the edges of G' the orientation that they have in G.

As in Section 4.4.2 we return two sets of cycles. The first set is constructed as follows. For each edge $e_i \in E \setminus E'$ for $1 \leq i \leq \lambda$ we compute the cycle $e_i + p_i$ where p_i is the shortest path in G' between the endpoints of e_i when G' is viewed as an undirected graph. Then, we traverse each such cycle in an arbitrary orientation and form our directed cycles based on the direction of the edges in G. The second set is simply the set of cycles of a directed MCB of G'. The proof of correctness of this approach is similar to the one presented in Section 4.4.2 if we replace the base field with $\mathbb{Q}$.

The time to compute our spanner is again $O(mn^{1+1/k})$. The fastest deterministic algorithm [45] to compute an MCB of a directed graph has a running time of $\tilde{O}(m^2N^{\omega-1} + N^3) + O(N(m^2n + mn^2 \log n))$. We execute this algorithm on our spanner and G' has at most $O(n^{1+1/k})$ edges. Thus, $N \leq m \in O(n^{1+1/k})$ and the running time becomes $O(mn^{1+1/k}) + \tilde{O}(n^{(1+1/k)(\omega+1)}) + O(n^{4+3/k})$. For $k \geq 1$ this is $O(n^{4+3/k})$.

Theorem 4.19. *A $(2k-1)$-approximate minimum cycle basis, for any integer $k \geq 1$, in a weighted directed graph can be computed in time $O(n^{4+3/k})$.*

If we allow the use of randomization then we can compute approximate directed MCB even faster. The Monte Carlo algorithm in [55] computes an MCB of a directed graph with positive weights in time $O(m^2n \log n)$ with high probability. Using this algorithm we get the following theorem.

Theorem 4.20. *A $(2k-1)$-approximate minimum cycle basis, for any integer $k \geq 1$, in a positively weighted directed graph can be computed with high probability in time $O(n^{3+2/k} \log n)$.*

4.7 Concluding Remarks

In this chapter we have obtained fast algorithms which compute approximate minimum cycle bases of graphs. We presented an α-approximate algorithm for any $\alpha > 1$ and several constant factor approximate algorithms. These algorithms are considerably faster than any current approach which computes exactly a minimum cycle basis.

The minimum cycle basis problem is used mostly as preprocessing and usually any sparse cycle basis suffices. Thus, our constant factor algorithms have high practical value. Moreover, one of our algorithms even when implemented without fast matrix multiplication runs in time $O(n^{3+3/k})$. We elaborate on this further in Section 5.7.

For sufficiently dense graphs our algorithms are $o(m^{\omega})$. Furthermore, we developed fast approximation algorithms for special classes of graphs. Our techniques extend to the directed minimum cycle basis problem.

Chapter 5

Minimum Cycle Basis Algorithms in Practice

Summary

In this chapter we consider the practical problem of computing a minimum cycle basis of an undirected graph $G = (V, E)$ with n vertices and m edges. We describe an efficient implementation of an $O(m^3 + mn^2 \log n)$ algorithm presented in [20]. For sparse graphs this is the currently fastest known algorithm. This algorithm's running time can be partitioned into two parts with time $O(m^3)$ and $O(m^2 n + mn^2 \log n)$ respectively. Our experimental findings imply that in random graphs the true bottleneck of a sophisticated implementation is the $O(m^2 n + mn^2 \log n)$ part. A straightforward implementation would require $N \cdot n$ single source shortest paths computations, thus we develop several heuristics in order to get a practical algorithm. Our experiments show that in random graphs our techniques result in a significant speedup.

Based on our experimental observations we combine the two fundamentally different approaches, which have been used so far to compute a minimum cycle basis, to obtain a new hybrid algorithm with running time $O(m^2 n^2)$. The hybrid algorithm is very efficient in practice for random dense unweighted graphs.

We also compare these two algorithms with a number of previous implementations for finding a minimum cycle basis of an undirected graph. Finally, we discuss the performance of our approximation algorithms in practice.

5.1 Introduction

In this chapter we consider the practical problem of computing a minimum cycle basis of an undirected graph $G = (V, E)$ with n vertices and m edges. We describe an efficient implementation of Algorithm 3.1, using LEDA [67]. For sparse graphs this is the currently fastest known algorithm. As discussed in Chapter 3, this algorithm's running time can be partitioned into two parts with time $O(m^3)$ and $O(m^2 n + mn^2 \log n)$ respectively. Our ex-

perimental findings imply that in random graphs the true bottleneck of a sophisticated implementation is the $O(m^2n + mn^2 \log n)$ part. A straightforward implementation would require $N \cdot n$ single source shortest paths computations, thus we develop several heuristics in order to get a practical algorithm. Our experiments show that in random graphs our techniques result in a significant speedup.

Based on these experimental observations, we combine the two fundamentally different approaches to compute a minimum cycle basis, to obtain a new hybrid algorithm with running time $O(m^2n^2)$. This new hybrid algorithm is very efficient in practice for random dense unweighted graphs.

We also compare these two algorithms with a number of previous implementations [53, 19] for finding a minimum cycle basis of an undirected graph. These implementations were developed in order to analyze complex electrical networks, more precisely to compute a sparse cycle basis to describe the 'voltage law' part of a system. Both contain algorithms which compute optimal or sub-optimal cycle bases. For our comparisons we used only the implementations which compute optimal cycle bases.

Finally, we discuss our approximation algorithms and their applicability in practice.

5.1.1 Experimental Setup

This chapter contains some experimental results and comparisons. All experiments are done using random sparse and dense graphs.

Random Graphs

The $G(n;p)$ model is due to Erdős and Rényi [29]. Let $G(n)$ be the set of all undirected graphs (without self-loops) on n vertices and define $M := \binom{n}{2}$; M is the number of edges in a complete graph on n vertices. $G(n)$ has precisely 2^M elements. In the $G(n;p)$ model each of the M potential edges is present with probability p, independently of the other edges. That is, the sample space of $G(n;p)$ is the entire set $G(n)$, and the probability of a graph $G \in G(n)$ with m edges is $p^m(1-p)^{M-m}$. If $p = \frac{1}{2}$ all graphs in $G(n)$ are equiprobable.

All graphs, which we use in our experiments, were constructed using the $G(n;p)$ model for $p = 4/n$, 0.3, and 0.5.

Platform[†]

Our implementation uses LEDA [67]. All experiments were performed on a Pentium M 1.7GHz machine with 1GB of memory and 1MB L2 cache, running GNU/Linux. We used the GNU g++ 3.3 compiler with the -O optimization flag. All other implementations considered use the boost C++ libraries [12].

5.2 Heuristics

In this section we present several heuristics which can improve the running time substantially. All heuristics preserve the worst-case time and space bounds.

5.2.1 Compressed and/or Sparse Representation (H1)

All vectors (sets S and cycles C) which are handled by the algorithm are in $\{0,1\}^m$ or $\{0,1\}^N$. Moreover, any operations performed are normal set operations. This allows us to use a compressed representation where each entry of these vectors is represented by a bit of an integer. This allows us to save up space and at the same time to perform 32 or 64 bitwise operations in parallel.

Except from a compressed representation we can also use a sparse representation. Every cycle C or set S is simply a collection of edges. Thus, we can represent such sets by a sorted list of integers. Each integer uniquely represents an edge. The only operation required is performing symmetric difference and this can be done in time proportional to the sizes of the vectors involved.

Which of the two above representations to use, seems to be relevant to the particular instances of the problem that we need to solve. For example if we know that the minimum cycle basis contains only cycles of small length, then a sparse representation seems more relevant. We believe that when solving small instances a compressed representation should be preferable but on larger instances a sparse one would provide better performance and less space consumption.

[†]The various implementations in this chapter use different graph libraries, either LEDA [67] or boost [12]. This could be the reason, for some additional constant factor performance differences between the algorithms.

5.2.2 Upper Bounding the Shortest Path (H2)

During phase i we might perform up to n shortest paths computations in order to compute the shortest cycle C_i with an odd intersection with the set S_i. We can thus use the shortest path found so far as an upper bound on the shortest path. This is implemented as follows; a node is only added in the priority queue of Dijkstra's implementation if its current distance is not more than our current upper bound.

5.2.3 Reducing the Shortest Path Computations (H3)

We come to the most important heuristic. In each of the N phases we are performing single source shortest paths computations between at most n pairs of vertices or one all pairs shortest paths computation. In the general case this can be accomplished by n single source shortest paths computations. This results in total to $N \cdot n$ single source shortest paths computations.

Let $S = \{e_1, e_2, \ldots, e_k\}$ be a *witness* at some point of the execution. We need to compute the shortest cycle C such that $\langle C, S\rangle = 1$. We can reduce the number of shortest paths computations based on the following observation. Let $C_{\geq i}$ be the shortest cycle in G such that:

(i) $\langle C_{\geq i}, S\rangle = 1$,

(ii) $C_{\geq i} \cap \{e_1, \ldots, e_{i-1}\} = \emptyset$, and

(iii) $e_i \in C_{\geq i}$.

Then, cycle C can be expressed as

$$C = \min_{i=1,\ldots,k} C_{\geq i} \ . \tag{5.1}$$

Note that at least one of the $C_{\geq i}$ for $1 \leq i \leq k$ exists, since we make sure that there is always at least one cycle with an odd intersection with S.

We can compute $C_{\geq i}$ in the following way. We delete edges $\{e_1, \ldots, e_i\}$ from the graph G and the corresponding edges from the signed graph G_i. Let $e_i = (v, u) \in G$. Then, we compute a shortest path in G_i from v^+ to u^+. The path computed will have an even number of edges from the set S and together with e_i an odd number. Since we deleted edges $\{e_1, \ldots, e_i\}$ the resulting cycle does not contain any edges from $\{e_1, \ldots, e_{i-1}\}$.

Using the above observation we can compute each cycle in $O(k \cdot \mathrm{SP}(n, m))$ time when $|S| = k < n$ and in $O(n \cdot \mathrm{SP}(n, m))$ when $|S| \geq n$. Thus, the

running time for the cycles computations is equal to

$$\mathrm{SP}(n,m) \cdot \sum_{i=1}^{N} min\{n, |S_i|\} \tag{5.2}$$

where $\mathrm{SP}(n,m)$ denotes the time to compute single source shortest paths in an undirected non-negative weighted graph with n vertices and m edges.

5.2.4 Basis Reordering (H4)

The main invariant of all algorithms based on the algebraic framework (Algorithm 3.2) which was presented in Section 3.2, is that for $1 \leq i \leq N$, S_i must be a vector in the subspace orthogonal to the cycles $\{C_1, \ldots, C_{i-1}\}$. Several vectors in this subspace suffice for the correctness of the MCB algorithm.

In Algorithm 3.1 at the beginning of phase $1 \leq i \leq N$ we have a set of vectors $\{S_i, \ldots, S_N\}$ which form a basis of the subspace orthogonal to the cycles $\{C_1, \ldots, C_{i-1}\}$. Heuristic *H3* implies that computing cycle C_i can be performed with a different number of shortest paths computations, based on the sparseness of the vector from the orthogonal subspace that is chosen. The *basis reordering* heuristic swaps S_i with S_j such that $i \leq j \leq N$ and $|S_j|$ is minimized.

The goal of this heuristic is to make a significant progress at the beginning of the algorithm. As the computation reaches its final stages, most of the orthogonal basis vectors will be relative dense and thus the heuristic will not have a big effect. Moreover, the dimension of the subspace orthogonal to $C_1, \ldots, C_{i-1}$ decreases during the algorithm. Thus, at the final stages we do not have a wide choice on which basis vector to choose. At early stages however, only a few shortest paths computations will be necessary in order to compute each MCB cycle.

This heuristic is less applicable to Algorithm 3.3 since in each phase we have much fewer S_i's in the subspace orthogonal to the set of cycles $\{C_1, \ldots, C_{i-1}\}$ that in Algorithm 3.1.

5.3 Updating the Witnesses, S_i's

In this section we present experimental results which suggest that the dominating factor of the running time of Algorithm 3.1 (at least for random graphs) is not the time needed to update the witnesses but the time to compute the cycles.

sparse ($m \approx 2n$)		
n	S's	C's
77	0.01	0.02
154	0.0100013	0.250001
303	0.0300028	0.770003
541	0.0500085	4.35001
795	0.230037	13.4
1060	0.519957	28.8599
1412	1.07985	59.2798
2070	2.97977	267.09
2505	4.46042	377.281
3336	9.84878	795.548

	$G(n; 0.3)$		$G(n; 0.5)$	
n	S's	C's	S's	C's
50	0.0100001	0.33	0.100001	0.66
70	0.0900026	1.44	0.339982	2.92998
100	0.529994	5.75	1.75995	15.0799
150	3.55005	37.6801	9.66977	114.75
200	12.5297	145.8	35.8403	421.82
250	34.578	508.309	99.888	1556.64

Table 5.1: Comparison of the time (in seconds) taken to update the sets S_i, $1 \leq i \leq N$ and the time taken to calculate the cycles C_i, $1 \leq i \leq N$ in random weighted graphs, by Algorithm 3.1.

Recall that the time to update the witnesses is $O(m^3)$ and the time to compute the cycles is $O(m^2n + mn^2 \log n)$. Thus, on sparse graphs Algorithm 3.1 has the same running time $O(n^3 \log n)$ as Algorithm 3.3. Algorithm 3.3 has a running time of $O(m^2n + mn^2 \log n + m^\omega)$; the m^ω factor is dominated by the m^2n but we present it here in order to understand what type of operations the algorithm performs. Recall that Algorithm 3.3 improves upon Algorithm 3.1 with respect to the time needed to update the sets S by using fast matrix multiplication.

Although fast matrix multiplication can be practical for medium and large sized matrices, our experiments show that in random graphs the time needed to update the S_i's is a small fraction of the time needed to compute the cycles. Table 5.1 presents a comparison of the required time to update the S_i's and to calculate the cycles C_i's by using the signed graph G_i for random weighted graphs.

In order to get a better understanding of this fact, we performed several experiments. As it turns out, in practice, the average cardinality of the S_i's is much less than N and moreover the number of times we actually perform set updates (if $\langle C_i, S_j \rangle = 1$) is much less than $N(N-1)/2$. Moreover, heuristic *H1* helps to further decrease the time required to perform these updates. Table 5.2 summarizes our results.

5.4 Number of Shortest Path Computations

Heuristic *H3* dramatically improves the best case of the Algorithm 3.1, while maintaining at the same time the worst case. Instead of $N \cdot n$ single source shortest paths computations, we perform much less. In Table 5.4 we study

n	m	N	$N(N-1)/2$	$max(\|S\|)$	$\lceil avg(\|S\|) \rceil$	$\#\ \langle S, C\rangle = 1$
sparse ($m \approx 2n$)						
10	19	10	45	4	2	8
51	102	52	1326	25	3	113
104	208	108	5778	44	4	258
206	412	212	22366	108	5	760
491	981	500	124750	226	7	2604
596	1192	609	185136	315	6	2813
963	1925	985	484620	425	7	5469
1060	2120	1084	586986	498	7	5980
1554	3107	1581	1248990	537	8	9540
2070	4139	2105	2214460	1051	13	20645
3032	6064	3092	4778686	1500	13	31356
4441	8882	4525	10235550	2218	17	58186
$G(n; 0.3)$						
10	13	4	6	2	2	2
25	90	66	2145	27	3	137
50	367	318	50403	133	5	1136
75	832	758	286903	370	6	3707
100	1485	1386	959805	613	7	8103
150	3352	3203	5128003	1535	9	22239
200	5970	5771	16649335	2849	10	49066
300	13455	13156	86533590	6398	10	116084
500	37425	36926	681746275	18688	14	455620
$G(n; 0.5)$						
10	22	13	78	7	2	14
25	150	126	7875	57	4	363
50	612	563	158203	298	6	2527
75	1387	1313	861328	654	6	6282
100	2475	2376	2821500	1168	8	15771
150	5587	5438	14783203	2729	9	39292
200	9950	9751	47536125	4769	11	86386
300	22425	22126	244768875	10992	13	227548
500	62375	61876	1914288750	30983	15	837864

Table 5.2: Statistics about sets S sizes on sparse random graphs with $p = 4/n$ and dense random graphs for $p = 0.3$ and 0.5. Sets are considered during the whole execution of Algorithm 3.1. Column $\#\langle S, C\rangle = 1$ denotes the number of updates performed on the S_i's. An upper bound on this is $N(N-1)/2$, which we actually use when bounding the algorithm's running time. Note that the average cardinality of the S_i's is very small compared to N although the maximum cardinality of some S_i's is in $O(N)$.

$G(n; 0.3)$		
n	H3 Heur.	NO Heur.
35	0.06	0.47
50	0.32	2.39
60	0.85	6.11
80	2.41	23.2
100	6.55	68.34
120	14.82	169.41
150	41.6	604.07

$G(n; 0.5)$		
n	H3 Heur.	NO Heur.
25	0.06	0.25
40	0.31	2.08
50	0.82	6.38
60	1.95	14.78
75	4.7	42.6
85	7.41	78.28
100	17.77	210.69
120	39.78	521.93
150	122.2	1954.24

sparse ($m \approx 2n$)		
n	H3 Heur.	NO Heur.
51	0.01	0.19
104	0.07	1.48
206	0.52	11.85
491	3.3	154.31
656	7.61003	392.29

Table 5.3: Running times in seconds of Algorithm 3.1 with and without the H3 *heuristic on random graphs. Without the heuristic the algorithm performs* $\Theta(nm)$ *single source shortest paths computations.*

the sizes of the sets S_i ($S_{i,i}$ in the terminology of Algorithm 3.1) for $i = 1, \ldots, N$ used to calculate the cycles for sparse and dense graphs respectively.

The main observation here is that in both sparse and dense graphs although the maximum set can have quite large cardinality (something like $O(N)$) the average set size is much less than n. Moreover, in sparse graphs every set used has cardinality less than n. On dense graphs the sets with cardinality less than n are more than 95% percent. Thus, heuristic *H3* is almost always in effect which in turn implies a significant speedup. Table 5.3 compares the running times of Algorithm 3.1 with and without the *H3* heuristic.

The heuristic for reducing the number of shortest paths computation has a dramatic effect on the performance of Algorithm 3.1. Nevertheless, the observation that for random graphs the shortest paths are the dominating factor of the algorithm remains valid. This fact is motivating enough, in order to try to design an algorithm which performs fewer shortest paths computations.

n	m	N	$max(\|S_i\|)$	$\lceil avg(\|S_i\|) \rceil$	$\|\{S_i : \|S_i\| < n\}\|$
sparse ($m \approx 2n$)					
10	19	10	4	2	10
51	102	52	16	5	52
104	208	108	39	5	108
206	412	212	106	10	212
491	981	498	246	13	498
596	1192	609	220	11	609
963	1925	980	414	11	980
1060	2120	1076	496	17	1076
1554	3107	1573	795	21	1573
2070	4139	2108	1036	27	2108
3032	6064	3092	1468	33	3092
4441	8882	4522	1781	33	4522
$G(n; 0.3)$					
10	13	4	2	2	4
25	90	66	20	4	66
50	367	318	153	10	302
75	832	758	357	15	721
100	1485	1386	638	15	1343
150	3352	3203	1534	18	3133
200	5970	5771	2822	29	5635
300	13455	13156	6607	32	12968
500	37425	36926	15965	39	36580
$G(n; 0.5)$					
10	22	13	7	3	13
25	150	126	66	5	121
50	612	563	222	12	532
75	1387	1313	456	10	1276
100	2475	2376	1094	15	2314
150	5587	5438	2454	19	5338
200	9950	9751	4828	28	9601
300	22425	22126	10803	33	21875
500	62375	61876	30877	38	61483

Table 5.4: Statistics about sets S_i sizes on sparse random graphs with $p = 4/n$ and dense random graphs for $p = 0.3$ and 0.5, at the moment we calculate cycle C_i with Algorithm 3.1. Compare the average cardinality of S_i with n.

5.5 Combining the Two Approaches

In this section we develop a new algorithm for the minimum cycle basis problem with running time $O(m^2n^2)$. The algorithm is a combination of the two main approaches that have been used to compute a minimum cycle basis. Its main motivation is to reduce the number of shortest paths computations. We begin by describing Horton's algorithm [52] in more detail.

5.5.1 Horton's Algorithm

Unlike the algorithms that we have seen so far, Horton's algorithm does not compute the cycles one by one but instead computes a superset of the MCB and then greedily extracts the MCB by Gaussian elimination. This superset contains $O(mn)$ cycles which are constructed in the following way.

For each vertex $w \in V$ and edge $e = (u, v) \in E$ construct the *candidate* cycle $C = \mathrm{SP}(w, u) + \mathrm{SP}(w, v) + (u, v)$ where $\mathrm{SP}(a, b)$ is the shortest path from a to b (see Figure 5.1). If these two shortest paths do not contain a vertex other than w in common then keep the cycle otherwise discard it. Let us call this set of cycles the *Horton set* and denote it with $\mathbb{HS}$. We next show that $\mathbb{HS}$ always contains an MCB.

For simplicity we will make the following assumption. We refer the interested reader to [52] for the treatment of the more general case.

Assumption 5.1. All shortest paths in graph G are unique.

The proof consists of a series of simple lemmata.

Theorem 5.1 (Horton [52])**.** *Let u and v be two vertices in G, and let $\mathbb{B}$ be a cycle basis of G. Let P denote a path from u to v. Any cycle $C \in \mathbb{B}$ that contains u and v can be exchanged for either a cycle that includes the path P, or a cycle that excludes one of u and v.*

Proof. Let P_1 and P_2 be the two paths in C joining u and v. Assume $P \neq P_1$ and $P \neq P_2$, for otherwise C does not need to be exchanged. Define $C_1 = P + P_1$ and $C_2 = P + P_2$. Then, C_1 and C_2 are cycles and $C = C_1 + C_2$. By Theorem 2.3, C can be exchanged for either C_1 or C_2. If the exchanged cycle C_i is not a simple cycle, then C_i can be exchanged for one of its simple cycles and that simple cycle cannot include both u and v. □

Corollary 5.2 (Horton [52])**.** *If $\mathbb{B}$ is a minimum cycle basis and P is the unique shortest path from x to y, then every cycle in $\mathbb{B}$ that contains x and y must contain the path P.*

Proof. In the proof of Theorem 5.1, $w(C_i) \leq w(P) + w(P_i) < w(P_1) + w(P_2) = w(C)$. Hence, if C does not contain P, it can be exchanged for a shorter cycle, a contradiction to the fact that $\mathbb{B}$ is a minimum cycle basis. □

Theorem 5.3 (Horton [52])**.** *Let w be any vertex of any cycle C in a minimum cycle basis of G. Then, there is an edge $(v, u) \in C$ such that C consists of a shortest path from w to v, a shortest path from w to u, and the edge (v, u).*

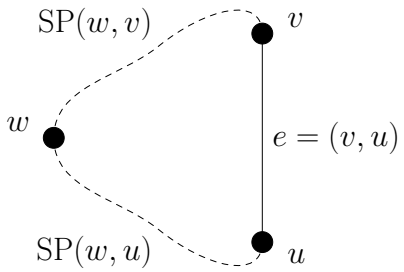


Figure 5.1: Candidate cycle for the MCB.

Proof. Let $C = (w_0, w_1, \ldots, w_{n-1}, w_n)$, where $w = w_0 = w_n$. Define $P_i = (w, w_1, w_2, \ldots, w_i)$ and $Q_i = (w, w_{n-1}, \ldots, w_i)$. By Corollary 5.2 and Assumption 5.1, $d(w, w_i) = w(P_i)$ or $w(Q_i)$. Let $v = w_i$, where i is the largest subscript such that $d(w, w_i) = w(P_i)$. Let $u = w_{i+1}$. Then, $C = P_i + (w_i, w_{i+1}) + Q_{i+1}$. □

However, not every MCB is contained in the Horton set. If we assume uniqueness of the MCB then it is immediately clear that the only MCB is contained in this set. Horton actually proved that if all edges have positive weights then it does not matter which shortest path is chosen between each pair of vertices.

Extracting an MCB from the Horton set is straightforward by using Gaussian elimination and the greedy algorithm. We first sort the cycles by length. The i-th cycle of the Horton set is in the MCB if it is linearly independent from the cycles which we have already selected among the first $i-1$ cycles of the Horton set. We continue this process until we have selected N such cycles. Note that this greedy step is really necessary.

5.5.2 A Hybrid Algorithm

The two approaches that we have seen so far for computing an MCB can be combined, and this is the purpose of this section. Both Algorithm 3.1 and Algorithm 3.3 require $O(mn(m+n\log n))$ time for the shortest paths queries. Our experimental observation was (recall Table 5.1) that for random graphs this is the dominating factor. Thus, it seems only natural to try to reduce this factor.

The main idea here is to exchange the shortest paths computations performed by Algorithm 3.1 and Algorithm 3.3 by the computation of $\mathbb{HS}$. The resulting algorithm has again N phases, one for each cycle of the MCB. During these phases we maintain the invariant that in phase i, S_i is a vector

Algorithm 5.1: Hybrid MCB algorithm

```
Ensure uniqueness of the shortest path distances of G (lexicographically or by perturbation)
Construct superset (Horton set) ℍ𝕊 of MCB
Let S_i = {e_i} for all i = 1, ..., N
for i = 1 to N do
    Find C_i as the shortest cycle in ℍ𝕊 s.t ⟨C_i, S_i⟩ = 1
    for j = i + 1 to N do
        if ⟨C_i, S_j⟩ = 1 then
            S_j = S_j + S_i
        end
    end
end
```

in the subspace orthogonal to the cycles $C_1, \ldots, C_{i-1}$. To compute the cycle C_i in phase i, we search for the shortest cycle $C \in \mathbb{HS}$ such that $\langle C, S_i \rangle = 1$.

The Horton set contains an MCB but not necessarily all the cycles that belong to any MCB. We resolve this difficulty by ensuring uniqueness of the MCB. We ensure uniqueness by ensuring uniqueness of the shortest path distances on the graph (either by perturbation or by lexicographic ordering). After the preprocessing step, every cycle of the MCB will be contained in the Horton set. This proves the correctness of the new algorithm. A succinct description can be found in Algorithm 5.1.

Running Time

Theorem 5.4. *Algorithm 5.1 solves the minimum cycle basis problem in time* $O(m^2n^2)$.

Proof. The time to compute the cycles is only $O(n^2m)$. To see this note that it is enough to build one shortest path tree for each node $v \in V$. This takes time $O(n(m + n \log n))$. Then, we can construct the $O(mn)$ candidate cycles in $O(n^2m)$ since each cycle might contain at most $O(n)$ edges.

Querying the Horton set takes time $O(m^2n^2)$ since the Horton set contains at most mn cycles, we need to search for at most m cycles and each cycle contains at most n edges. □

The space requirement is $O(mn^2)$, a factor of n more than $O(mn)$ which is the space required to represent the cycles of the MCB. The important property of this algorithm is that the time to actually compute the cycles is only $O(n^2m)$. This is a factor of $\frac{m}{n}$ better than the $O(m^2n)$ time required by

Algorithm 3.1 or Algorithm 3.3. Together with the experimental observation that in general the linear independence step is not the bottleneck, we actually hope to have developed an efficient algorithm.

5.6 Running Times Comparison

In this section we compare various implementations for computing a minimum cycle basis. We include in our comparison the following algorithms:

(i) our implementation [65] of Algorithm 3.1 denoted as `DP` or `DP_U` in the case of unweighted graphs,

(ii) our implementation of Algorithm 5.1 denoted as `HYB` or `HYB_U` in the case of unweighted graphs,

(iii) an implementation [19] of Horton's algorithm denoted as `HOR` in case of weighted and as `HOR_U1` in case of unweighted graphs,

(iv) another implementation [53] of Horton's algorithm, for unweighted graphs, denoted as `HOR_U2`, and

(v) an implementation [19] of another $O(m^3 + mn^2 \log n)$ algorithm due to Berger et al. [8], which is similar to de Pina's algorithm, denoted as `FEAS` for weighted and `FEAS_U` for unweighted graphs.

Implementations (iii) and (iv) contain some common codes. We include both in the comparison because there seems to be a constant factor difference in the running times. Fast matrix multiplication can nicely improve many parts of these implementations with respect to the worst case complexity. We did not experiment with these versions of the algorithms.

The comparison of the running times is performed for three different type of undirected graphs:

(a) random sparse graphs, where $m \approx 2n$,

(b) random graphs from $G(n; p)$ with different density $p = 0.3, 0.5$, and

(c) hypercubes.

Tests are performed for both weighted and unweighted graphs. In the case of weighted graphs the weight of an edge is an integer chosen independently at random from the uniform distribution in the range $[0 \dots 2^{16}]$.

Figure 5.2 and Figure 5.3 contain the results of the comparisons for unweighted and weighted graphs respectively. In the case of weighted graphs

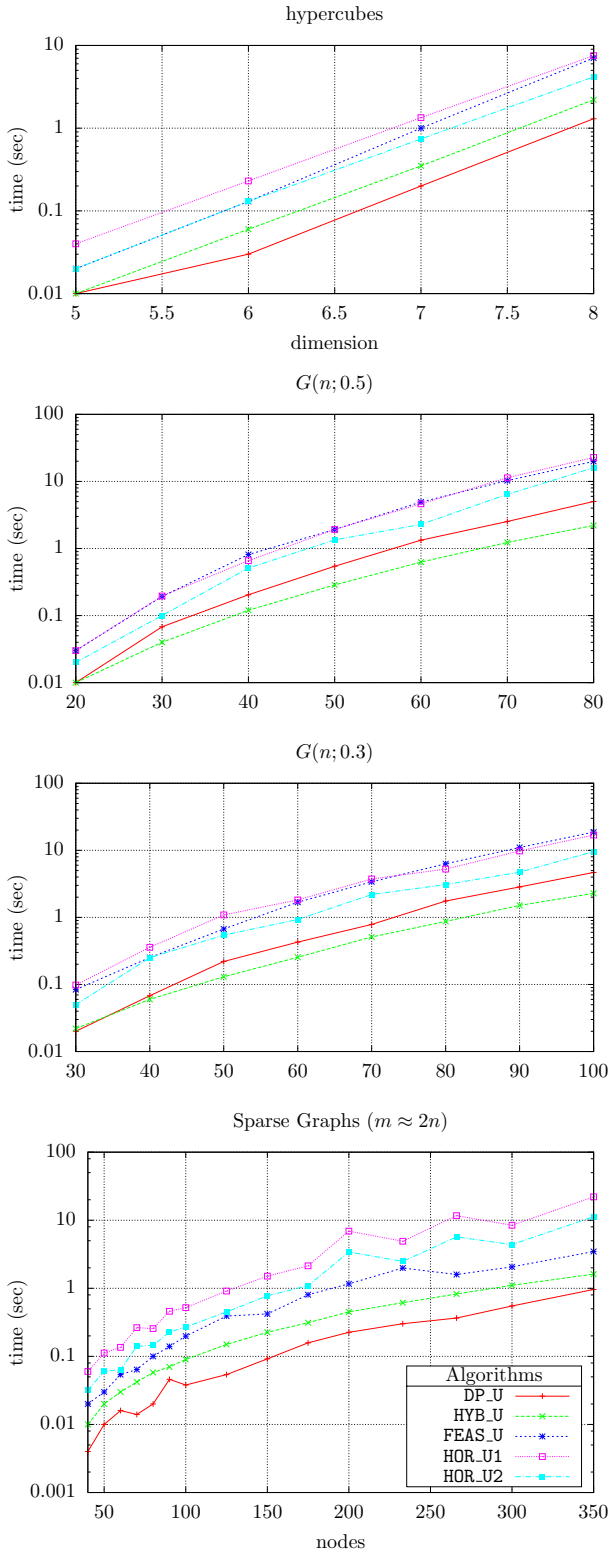


Figure 5.2: Comparison of various algorithms for random unweighted *graphs. Algorithm 3.1 is denoted as* `DP_U` *and Algorithm 5.1 as* `HYB_U`. `HOR_U1` *and* `HOR_U2` *are two different implementations of Horton's algorithm.* `FEAS_U` *is an implementation of another* $O(m^3)$ *algorithm. See Section 5.6 for details.*

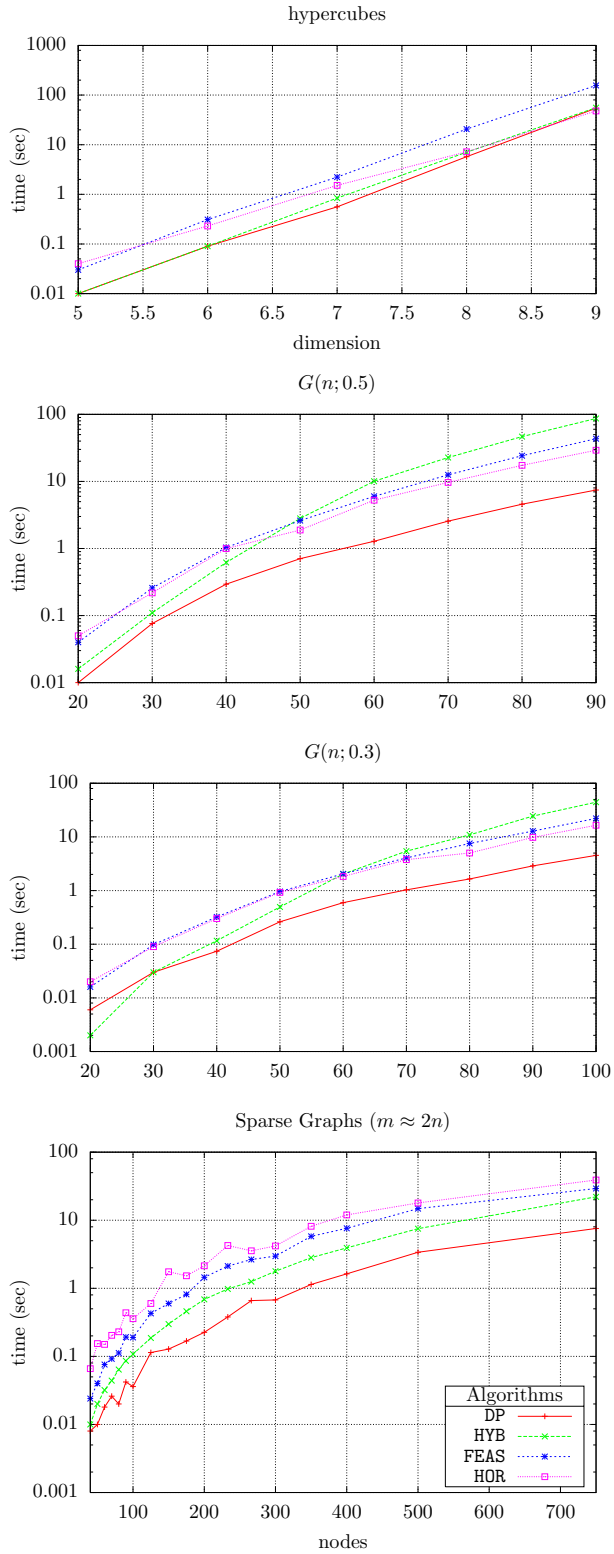


Figure 5.3: Comparison of various algorithms for random weighted *graphs. Algorithm 3.1 is denoted as* `DP` *and Algorithm 5.1 as* `HYB`. `HOR` *is Horton's algorithm and* `FEAS` *is another* $O(m^3 + mn^2 \log n)$ *algorithm. See Section 5.6 for details.*

Algorithm 3.1 outperforms all the other algorithms. It is also worth noting that the hybrid algorithm (Algorithm 5.1) outperforms Horton's algorithm in the case of sparse graphs but it is slower in all other cases. On the other hand in the case of unweighted graphs, except for sparse graphs and hypercubes where Algorithm 3.1 is superior, for dense graphs Algorithm 5.1 outperforms all the remaining algorithms. As can be easily observed the differences on the running time of the implementations are rather small for sparse graphs. For dense graphs however, we observe a substantial difference in performance.

5.6.1 Dense Unweighted Graphs

In the case of dense unweighted graphs, the hybrid algorithm performs better than the other algorithms. However, even on the exact same graph, the addition of weights changes the performance substantially. This change in performance is not due to the difference in size of the produced Horton set between the unweighted and the weighted case, but due to the total number of queries that have to be performed in this set.

In the hybrid algorithm before computing the MCB, we sort the cycles of the Horton set. Then, for each of the N phases we *query* the Horton set, from the least costly cycle to the most, until we find a cycle with an odd intersection with our current witness S. Figure 5.4 presents for dense graphs the number of cycles in the Horton set (the size of the Horton set is considered with duplicate cycles) and the number of queries required in order to extract the MCB from this set. In the case of unweighted graphs, the number of queries is substantially smaller than in the case of weighted graphs. This is exactly the reason why the hybrid algorithm outperforms the others in unweighted dense graphs.

5.7 Approximation Algorithms

In this section we consider the approximation algorithms, Algorithm 4.2 and Algorithm 4.3. Both algorithms use fast matrix multiplication. However, they are also efficient even when used without fast matrix multiplication. This fact has high practical value since high performance fast matrix multiplication libraries are difficult to implement.

Concerning Algorithm 4.3, the fastest algorithm to compute exactly an MCB without fast matrix multiplication is Algorithm 3.1 and requires $O(m^3 + mn^2 \log n)$ time. Since we only execute the exact algorithm in the spanner where $m \in O(n^{1+1/k})$ we get the following theorem.

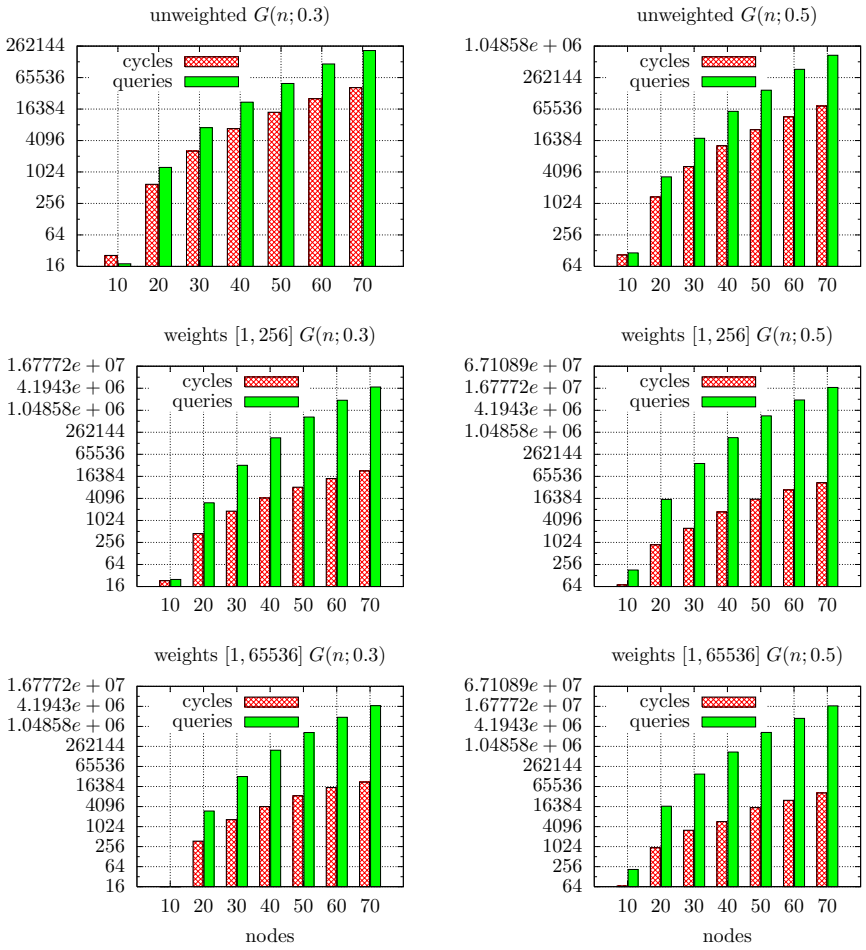


Figure 5.4: Number of cycles in the Horton set (set with duplicates) and number of queries required in this set (set sorted by cycle weight) in order to extract the MCB for random dense graphs with random weights of different ranges. Each random graph is considered with three different edge weight ranges: (a) unweighted, (b) weights in $[1, 2^8]$, (c) weights in $[1, 2^{16}]$.

Theorem 5.5. *Algorithm 4.3 computes a $(2k-1)$-approximate minimum cycle basis, for any integer $k \geq 1$, in an undirected weighted graph without fast matrix multiplication in time $O(n^{3+3/k})$.*

Algorithm 4.2 can also be implemented without fast matrix multiplication. The last phase is the only part which depends on fast matrix multiplication techniques. Instead of using Algorithm 3.3 we use Algorithm 3.1. The running time of this approach is $O(m\mu^2) = O(mn^{2+2/k})$

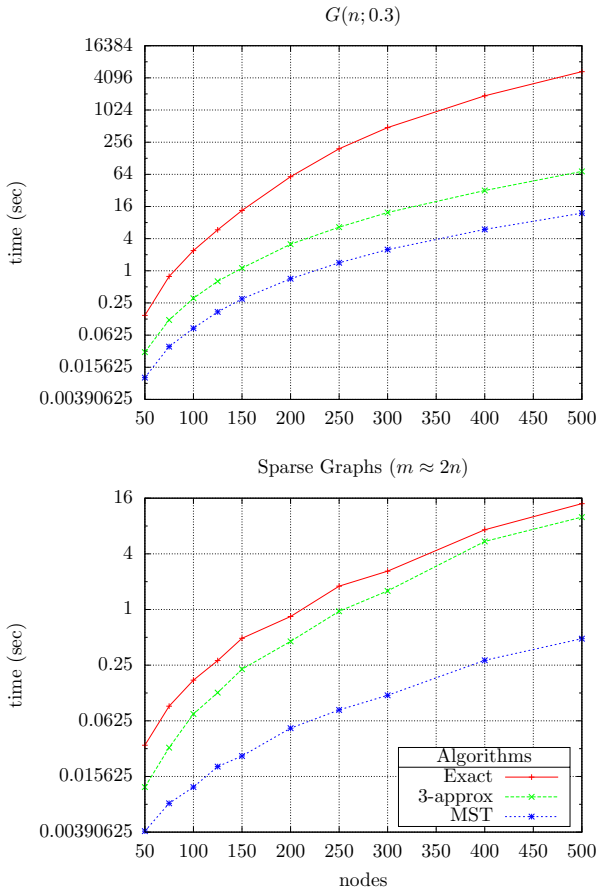


Figure 5.5: Performance of the $(2k-1)$-approximation Algorithm 4.3 without fast matrix multiplication (see Theorem 5.5) in random weighted graphs. Three variants are used $k = 1$ (exact computation), $k = 2$ (3-approximation) and $k = \infty$ (minimum spanning tree fundamental basis).

plus $O(\mu(kn + kmn^{1/k}))$ for the shortest paths. We have shown the following theorem.

Theorem 5.6. *Algorithm 4.2 computes a $(2k-1)$-approximate minimum cycle basis, for any integer $k \geq 1$, in an undirected weighted graph without fast matrix multiplication in expected time $O(kmn^{1+2/k} + mn^{2+2/k})$.*

Both algorithms are $o(m^\omega)$ for sufficiently dense graphs and appropriate values of k. Moreover, they are easy to implement efficiently.

Figure 5.5 presents the performance of Algorithm 4.3 without fast matrix multiplication. We again use random weighted graphs (weights in $[1, 2^{16}]$) in our experiments and use three variants of the algorithm. The first is an exact computation using $k = 1$. The second is a 3-approximation algorithm using $k = 2$ and the last is using $k = n$ (same behavior as $k = \infty$). When k equals the number of nodes of the graph the spanner computation returns the minimum spanning tree and thus the cycle basis returned is the fundamental

cycle basis induced by the minimum spanning tree. The algorithm in this case needs time $O(mn)$ but the approximation factor is $O(n)$. We present it in the figure for comparison.

As we expected, the running time of the 3 approximation algorithm is always better than the running of the exact algorithm. For sparse graphs there is a small improvement. On the other hand for dense graphs the performance of the 3 approximation algorithm is significantly better than the exact approach. Recall that the goal of Algorithm 4.3 is to reduce the MCB computation of a dense graph to a computation in a sparser graph. As we increase k the performance is getting better until it reaches the time bound of the algorithm which constructs the fundamental minimum spanning tree basis.

5.8 Concluding Remarks

The purpose of this chapter was to study the minimum cycle basis problem from a practical point of view. To discover whether algorithms for the MCB problem can be implemented efficiently and whether any heuristic can be used to speed up these algorithms.

All algorithms for computing an MCB use some shortest paths subroutine. Our main observation, about random sparse and dense graphs, is that the shortest paths routines are the bottleneck of the algorithms. Thus, in this case any theoretical improvement using fast matrix multiplication (like the one presented in Chapter 3) in order to ensure linear independence is not necessary. Of course there are other graphs where this is not necessarily the case. One such example will be examined in more detail in Chapter 6.

The number of shortest paths computations seems to be rather important for the running time of MCB algorithms. We were unable however to prove some better upper bound on the number of shortest path computations necessary to compute an MCB. Currently the algorithm in Chapter 3 requires $O(mn)$ single source shortest paths or $O(m)$ all pairs shortest paths computations. We leave it as an open problem whether the method in Section 3.3 can be improved. Any improvement will immediately give a faster algorithm for the MCB problem. Such an improvement could be obtained with respect to the properties of the witnesses (for example the density of the S_i's) or completely independent of them.

A first step in this direction has been obtained in Chapter 4 but in the expense of returning an approximate solution.

Chapter 6

Sampled Manifolds

Summary

Point samples of a surface in $\mathbb{R}^3$ are the dominant output of a multitude of 3D scanning devices. The usefulness of these devices rests on being able to extract properties of the surface from the sample. We show that, under certain sampling conditions, the minimum cycle basis of a nearest neighbor graph of the sample encodes topological information about the surface and yields bases for the trivial and non-trivial loops of the surface. We validate our results by experiments.

6.1 Introduction

Point samples of a surface in $\mathbb{R}^3$ are the dominant output of a multitude of 3D scanning devices. The usefulness of these devices rests on being able to extract properties of the surface from the sample. A common approach into treating point samples is to first build some kind of neighborhood graph. The k-nearest neighbor graph is relatively easy to compute [5] and is a popular starting point for many algorithms extracting information about a surface.

Let S be a compact manifold in $\mathbb{R}^3$ and let P be a finite set of points in S. We call P a *point sample* of S. For an integer k, the k-nearest neighbor graph $G_k = G_k(P)$ on P is an undirected graph with vertex set P. It contains an edge between sample points a and b if b is one of the k points closest to a and a is one of the k points closest to b. Some researchers define G_k in an unsymmetric way by requiring that either b is k-closest to a or a is k-closest to b. We do not know whether our results apply to the alternative definition.

The main result of this chapter is as follows: We show that if S is a smooth surface and P a sufficiently dense sample of S, the minimum cycle

basis of the nearest neighbor graph of P gives information about the genus of S and yields bases for the set of trivial and non-trivial loops. Our experiments suggest that this is true also for some non-smooth surfaces.

More formally, we show that for suitably nice samples of smooth manifolds of genus g and sufficiently large k, the k-nearest neighbor graph G_k has a cycle basis consisting only of *short* (= length at most $2(k+3)$) and *long* (= length at least $4(k+3)$) cycles. Moreover, the MCB is such a basis and contains exactly $m-(n-1)-2g$ short cycles and $2g$ long cycles. The short cycles span the subspace of trivial loops and the long cycles form a homology basis; see next section or Section 2.3 for a definition. Thus, the MCB of G_k reveals the genus of S and also provides a basis for the set of trivial cycles and a set of generators for the non-trivial cycles of S.

We prove the statement for smooth manifolds, sufficiently dense samples, and sufficiently large k. A dense sample is defined in a manner similar to that of Amenta and Bern [2], as follows: The *medial axis* of a manifold S embedded in $\mathbb{R}^3$ is the closure of the set of points in $\mathbb{R}^3$ with more than one nearest neighbor on S. The *local feature size* $f : S \mapsto \mathbb{R}$ assigns to every point in S its least distance to the medial axis of S. The point sample P is called an ϵ-sample of S, if every point $x \in S$ has a point in P at distance at most $\epsilon f(x)$. If in addition $||p-q|| \geq \delta f(p)$ for all distinct $p, q \in P$ the sample P is called an (ϵ, δ)-sample. This definition requires the sample to be dense with respect to the local feature size, but at the same time samples cannot be arbitrarily close. Our main result is now as follows: If P is an (ϵ, δ)-sample of a smooth manifold for sufficiently small ϵ and k is sufficiently large (the meaning of sufficiently large depends on ϵ and δ), the claim of the preceding paragraph is true. The claim is also true if we use a weaker sampling assumption, namely *locally uniform* ϵ-samples [35].

6.1.1 Other Approaches and Remarks

Given a point sample, undoubtedly, the ultimate application is to form a geometric approximation of the surface based on the sample. The objective is typically to form a piecewise linear surface (i.e. triangle mesh) whose geometry and topology are as similar to the original as possible. For smooth surfaces and sufficiently dense samples, good and efficient reconstruction algorithms are available, see e.g., the papers [9, 51, 2, 3, 35, 24, 62] and the survey by Dey [22]. For non-smooth surfaces or sparse samples the problem is open.

The best asymptotic running time in order to do smooth surface reconstruction is $O(n \log n)$ [35] and efficient implementations are available [24,

23]. Certainly, once this is done, it is relatively easy to extract less detailed information such as the genus of the surface or a homology basis. For example, a homology basis of a manifold 3D mesh may be computed in $O(n)$ time by an algorithm of Vegter and Yap [80], and a *shortest* homology basis may be computed in $O(n^2 \log n + n^2 g + n g^3)$ by an algorithm of Erickson and Whittlesey [30].

Thus, in the case of smooth manifolds computing an MCB in order to compute less detailed information is an overkill. The purpose of this chapter is not to compete algorithmically with the previously mentioned approaches, but instead to study the properties and the combinatorial structure that the minimum (and some sparse) cycle bases of certain neighborhood graphs possess.

6.2 Structure of Cycles

6.2.1 The Basic Idea

Let S be a compact 2-manifold of genus g with no boundary embedded in $\mathbb{R}^3$. For the sequel we will need to define a number of topological concepts. Rather than give a complete formal exposition, which may be found in most algebraic topology textbooks (e.g. [68]), we provide a more intuitive informal set of definitions. See also Chapter 2.

A closed simple curve on S is called a *loop*. The elements of the *first homology group* of S are equivalence classes of loops. The identity element of this homology group is the equivalence class of *separating loops*, namely, loops whose removal disconnects the surface. Two homology loops are in the same *homology class* if one can be continuously deformed into the other via a deformation that may include splitting loops at self-intersection points, merging intersecting pairs of loops, or adding and deleting separating loops. A loop is *trivial* if it is a separating loop and is *non-trivial* otherwise. A homology basis of S is any set of $2g$ loops whose homology classes span all non-trivial loops.

Let P be a sample of S and G_k its k-nearest neighbor graph. For any edge (a, b) of G_k we define a curve in S connecting a and b. Let us parameterize the segment ab by length. Let $p(t) = a + t(b - a)$ be the point with parameter value t, $0 \leq t \leq 1$ and let $q(t)$ be a point on S nearest to $p(t)$. *We assume that $q(t)$ is unique.* Then, $q(t)$ traces a continuous curve on S connecting a and b, which we denote by γ_{ab}. In this way, we obtain a drawing of G_k on S and cycles in G_k *induce* loops in S. In particular, cycles can induce trivial or non-trivial loops, in which case we call them trivial or non-trivial

cycles, respectively. The assumption that $q(t)$ is unique is not very stringent. Observe that if $q(t)$ is not unique, $p(t)$ is a point on the medial axis and hence $\max(f(a), f(b)) \leq ||a-b||/2$.

We next give general conditions under which every minimum cycle basis of G_k contains exactly $2g$ long cycles, the long cycles induce a homology basis of S over the reals, and all short (= non-long) cycles are trivial.

1. $L(\gamma_{ab}) \leq c_1||a-b||$ where c_1 is a constant, i.e., the length of the curve γ_{ab} is not much larger than the length of the segment ab.
2. There is a subgraph M of G_k (all vertices and a subset of the edges) such that
   - the drawing restricted to M is an embedding,
   - M is a mesh of genus g for S,
   - the faces of M have bounded size, say at most c_2 edges, and
   - for every edge $e \in G_k \setminus M$ the shortest path in M connecting a and b has bounded length, say bounded by c_3.
3. The minimal length of a non-trivial loop of S is L_{min} and for every edge (a,b) of the graph G_k the distance between its endpoints is at most $L_{min}/(2\max(c_2, c_3+1))$.

Theorem 6.1. *If the conditions above hold, every MCB of G_k contains exactly $m-(n-1)-2g$ short (length less than $\max(c_2, c_3+1)$) and exactly $2g$ long (length at least $2\max(c_2, c_3+1)$) cycles. Moreover, the long cycles induce a basis of the first homology group of S over the reals.*

Proof. The embedding of M has m_M edges and f faces where $f - m_M + n = 2-2g$. Consider the following set $\mathbb{B}$ of cycles: all face cycles of M but one and for each edge $e=(a,b) \in G_k \setminus M$ the cycle consisting of e plus the shortest path in M connecting a and b. Any cycle in $\mathbb{B}$ has length at most $\max(c_2, c_3+1)$ and the cycles in $\mathbb{B}$ are independent. There are $f - 1 + m - m_M = m_M - n + 2 - 2g - 1 + m - m_M = m - (n-1) - 2g$ cycles in $\mathbb{B}$.

Any cycle basis of G_k must contain at least $2g$ non-trivial cycles and these cycles induce loops which span the homology group of S. Consider any non-trivial cycle. It has length at least L_{min}. For any edge (a,b) of G_k the length of γ_{ab} is at most $L_{min}/(2\max(c_2, c_3+1))$ and hence any non-trivial cycle must contain at least $2\max(c_2, c_3+1)$ edges.

We have now shown that there are $m-(n-1)-2g$ independent cycles of length at most $\max(c_2, c_3+1)$ and that every non-trivial cycle consists

of at least $2\max(c_2, c_3 + 1)$ edges. Consider now any MCB $\mathbb{B}^*$ of G_k. It must contain at least $2g$ long cycles. Assume that it contains less than $m-(n-1)-2g$ short cycles. Then, at least one cycle in $\mathbb{B}$, call it C, is not spanned by the short cycles in $\mathbb{B}^*$, i.e., the representation of C as a sum of cycles in $\mathbb{B}^*$ contains a cycle D which is not short. Thus, we can improve the total length of B^* by replacing D by C. □

In Sections 6.2.3 and 6.2.4 we substantiate the theorem for smooth surfaces. We will actually prove a stronger result. Note that in the previous theorem we did not use the first condition. This condition will be useful later on when we will also replace the global condition of maximal edge length (see the third condition) by a local condition depending on the local feature size.

6.2.2 Sampling and Restricted Delaunay Triangulations

The local feature size is 1-Lipschitz continuous, i.e.,

Lemma 6.2 (Amenta and Bern [2]). *For any two points p and q on S, $|f(p) - f(q)| \leq ||p - q||$.*

Let D_P and V_P denote the Delaunay and the Voronoi diagram of P. The Voronoi cell $V_p \subset V_P$ for a point $p \in P$ is defined as the set of points $x \in \mathbb{R}^3$ such that $||x-p|| \leq ||q-x||$ for any $q \in P$ and $q \neq p$. The Delaunay triangulation of P is the dual of V_P. It has an edge pq if and only if V_p, V_q share a face, has a triangle pqr if and only if V_p, V_q, V_r share an edge, and a tetrahedron $pqrs$ if and only if V_p, V_q, V_r, and V_s share a Voronoi vertex. We assume that the input sample $P \in \mathbb{R}^3$ is in general position and that no vertex of V_P lies on S.

Consider the restriction of the Voronoi diagram V_P to the surface S. This defines the *restricted Voronoi diagram* $V_{P|S}$, with *restricted Voronoi cells* $V_{p|S} = V_p \cap S$. It is said to satisfy the *ball* property if each $V_{p|S}$ is a topological 2-ball, each nonempty pairwise intersection $V_{p|S} \cap V_{q|S}$ is a topological 1-ball, and each nonempty triple intersection $V_{p|S} \cap V_{q|S} \cap V_{r|S}$ is a single point.

The dual of the restricted Voronoi diagram defines the *restricted Delaunay triangulation* $D_{P|S}$. In more detail, an edge pq is in $D_{P|S}$ if and only if $V_{p|S} \cap V_{q|S}$ is nonempty and a triangle pqr is in $D_{P|S}$ if and only if $V_{p|S} \cap V_{q|S} \cap V_{r|S}$ is nonempty. Our general position assumption guarantees that there is no tetrahedron in $D_{P|S}$. It is known that the restricted Delaunay edges for an ϵ-sample are short.

Lemma 6.3 (Amenta and Bern [2], see also Giesen and Wagner [38]). *Let P be an ϵ-sample of S with $\epsilon < 1$. If pq is an edge of the restricted Delaunay triangulation, then $||p-q|| \leq \frac{2\epsilon}{1-\epsilon} min\{f(p), f(q)\}$.*

For $\epsilon \leq 0.08$, the restricted Voronoi diagram is known to have the ball property [2]. In this case the restricted Delaunay triangulation is a simplicial surface homeomorphic to S [28]. The k-neighborhood graph G_k contains the restricted Delaunay triangulation if P is a sufficiently nice sample of S.

Lemma 6.4 (Andersson et al.[4]). *Let P be an (ϵ, δ)-sample of S, let $w = \frac{2\epsilon}{1-\epsilon}$ and $k \geq \frac{(\delta(1+w)+w)^2}{\delta^2(1-w)^2-w^4}$. Then, the restricted Delaunay triangulation $D_{P|S}$ is a subgraph of G_k.*

Funke and Ramos [35] have shown how to turn any ϵ-sample into a *locally uniform* ϵ-sample by using decimation. Locally uniform samples are related to (ϵ, δ)-samples, but technically more involved. Both types of sample have the property that each sample point has only a constant number of restricted Delaunay neighbors. This is the only property that we require and thus, our results are valid for both cases. We have chosen to state our results in terms of (ϵ, δ)-samples, for ease of exposition.

We also state one more useful fact which we need later on. At each point $p \in S$, there are two tangent *medial balls* centered at points of the medial axis. The vectors from p to the centers of its medial balls are normal to S, and S does not intersect the interiors of the medial balls. Since $f(p)$ is at most the radius of the smaller medial ball, S is also confined between the two tangent balls of radius $f(p)$.

6.2.3 Short Cycles

In this section we show that for an appropriate sampling density there exists a linearly independent set of $m - (n-1) - 2g$ short cycles.

Warm-Up: The Planar Case

Consider a finite set P of points in the plane, its nearest neighbor graph G_k, its Delaunay triangulation D_P, and assume that D_P is contained in G_k. We describe a cycle basis consisting only of short cycles. In the next section, we will generalize the approach of this section to manifolds in $\mathbb{R}^3$.

Consider the following set $\mathbb{B}$ of cycles. It contains (a) the face cycles of all bounded faces of the Delaunay triangulation and (b) for each edge $e = (a, b) \in G_k \setminus D_P$ the cycle formed by e and the shortest path from a to b in D_P.

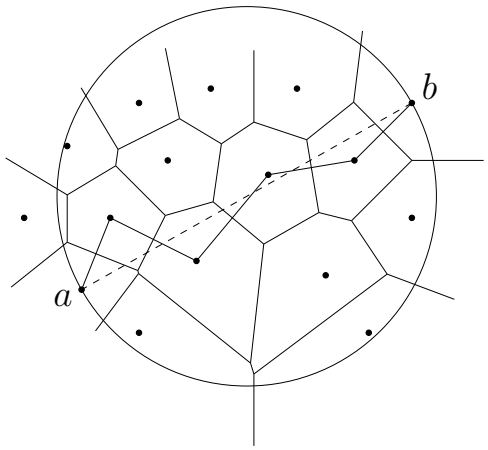


Figure 6.1: An induced path from a to b formed with edges of the triangulation.

Lemma 6.5. *$\mathbb{B}$ is a cycle basis and any cycle in $\mathbb{B}$ has length at most $k+1$.*

Proof. First we show that we have the right number of cycles. Let m_p be the number of edges of D_P and let m_k be the number of remaining edges. The dimension of the cycle space is $N = m - n + 1 = m_p + m_k - n + 1$. D_P is a planar graph and therefore $m_p - n + 1 = f - 1$ where f is the number of faces. Thus, $N = f - 1 + m_k = |\mathbb{B}|$.

The bounded faces of D_P are clearly linearly independent. The remaining cycles are also linearly independent since each of them contains an edge which is not contained in any other cycle in $\mathbb{B}$. This proves the first part of our lemma.

We come to the second part. Bounded faces of D_P have length 3. Consider next an edge $e = (a, b) \in G_k \setminus D_P$. The straight line segment ab from a to b crosses some cells of the Voronoi diagram of P and induces a path $a = b_0, b_1, \ldots, b_{l-1}, b_l = b$ in D_P, namely the path through the sites owning these cells, see Figure 6.1. This path is entirely contained in the circle with ab as its diameter [27].

It remains to show that this cycle is short. Since $e = (a, b)$ is an edge of G_k, there can be at most $k - 1$ other points in this circle and hence any cycle in $\mathbb{B}$ has length at most $k + 1$. □

We emphasize that this cycle basis is not necessarily the minimum cycle basis of the graph.

The 3-dimensional Case

We consider essentially the same set of cycles as in the preceding section: (a) all but one faces of the restricted Delaunay triangulation, and (b) for each remaining edge $e = (a, b) \in G_k \setminus D_{P|S}$ the cycle consisting of e plus the shortest path from a to b in the restricted Delaunay triangulation. As in the planar case, these cycles are linearly independent. It remains to prove that they are short. In this section we will prove the following theorem.

Theorem 6.6. *Let P be an ϵ-sample of S, let $k < \log_{\frac{1+\epsilon}{1-\epsilon}} \frac{3}{2}$, and assume that $D_{P|S} \subseteq G_k$. Let $(a, b) \in G_k \setminus D_{P|S}$. Then, there is a path from a to b in $D_{P|S}$ of length at most $2k + 5$.*

Consider the edge $e = (a, b) \in G_k \setminus D_{P|S}$, then the curve γ_{ab} crosses some cells of the restricted Voronoi diagram and induces a path $p_{ab} = (a = b_0, b_1, \ldots, b_{l-1}, b_l = b)$ in the restricted Delaunay triangulation, namely the path through the sites owning these cells. Since G_k contains the restricted Delaunay triangulation, p_{ab} exists in G_k.

Lemma 6.7. *Let $e = (a, b) \in G_k \setminus D_{P|S}$ and let B be the ball with ab as its diameter. If the induced path p_{ab} is contained in B, then the cycle consisting of e plus p_{ab} has length at most $k + 1$.*

Proof. Since $e \in G_k$, the ball B contains at most $k - 1$ other points apart from a and b. □

The above lemma generalizes the planar case. Unfortunately, we are unable to prove that p_{ab} runs within B. We therefore argue somewhat differently. We first show that either there is a very short path in $D_{P|S}$ from a to b or both a and b are far away from the medial axis. In the latter case we show that p_{ab} is contained in a slightly bigger ball but still sufficiently small for our purposes. We begin with the following auxiliary lemma.

Lemma 6.8. *Let a and b be two points in P and assume that there is a path of length l in $D_{P|S}$ from a to b. Let $\alpha = (1 + \epsilon)/(1 - \epsilon)$. Then, $\alpha - 1 = 2\epsilon/(1 - \epsilon)$ and*

$$||a - b|| \leq (\alpha^l - 1) \min(f(a), f(b)) \ .$$

Proof. We use induction on the length of the path. Let the path from a to b in $D_{P|S}$ be $a = q_0, q_1, \ldots, q_l = b$. Then, $||a - b|| = ||a - q_l||$.

For the base case $l = 1$ we have that $||a-q_1|| \leq (\alpha-1)f(a)$ by Lemma 6.3. Assume that the statement holds for paths of length $l-1$, then

$$\begin{aligned}
||a-b|| &= ||a-q_l|| \\
&\leq ||a-q_{l-1}|| + ||q_{l-1}-q_l|| \\
&\leq ||a-q_{l-1}|| + (\alpha-1)f(q_{l-1}) && \text{by Lemma 6.3} \\
&\leq ||a-q_{l-1}|| + (\alpha-1)(f(a)+||a-q_{l-1}||) && \text{by Lemma 6.2} \\
&= (\alpha-1)f(a) + \alpha||a-q_{l-1}|| \\
&\leq (\alpha-1)f(a) + \alpha(\alpha^{l-1}-1)f(a) && \text{by induction} \\
&= (\alpha^l-1)f(a)\ .
\end{aligned}$$

The same argument applies to b instead of a. □

Lemma 6.9. *Let $(a,b) \in G_k$, $\lambda \geq 1$ and assume that*

$$k < \log_{\frac{1+\epsilon}{1-\epsilon}} \frac{1+\lambda}{\lambda}\ . \tag{6.1}$$

Then, either there is a path of length at most k from a to b in the restricted Delaunay triangulation or

$$f(a), f(b) > \lambda||a-b||\ . \tag{6.2}$$

Proof. Recall that $\alpha = \frac{1+\epsilon}{1-\epsilon}$. Thus, inequality (6.1) implies $\alpha^k - 1 < 1/\lambda$. Consider the shortest path in the restricted Delaunay triangulation $a = q_0, q_1, \ldots, q_{l-1}, q_l = b$ from a to b. Furthermore, assume $l > k$ and $f(a) \leq \lambda||a-b||$. Lemma 6.8 implies that

$$||a-q_i|| \leq (\alpha^i-1)f(a) \leq (\alpha^k-1)f(a) < f(a)/\lambda \leq ||a-b|| \quad \text{for } 1 \leq i \leq k,$$

and hence there are k points closer to a than b is, a contradiction. The argument works symmetrically for b. □

The above lemma states that it is enough to prove Theorem 6.6 when $f(a), f(b) > \lambda||a-b||$. From now on, we proceed under this assumption for some $\lambda \geq 1$. Let us parameterize the segment ab by length. Let $p(t) = a+t(b-a)$ be the point with parameter value t, $0 \leq t \leq 1$ and let $q(t)$ be the point on S nearest to $p(t)$. Note that $q(t)$ is unique, because otherwise $p(t)$ would be a point of the medial axis contradicting the fact that $f(a) > ||a-b||$. Finally, let c denote the mid-point of the segment ab and $s(t)$ denote the site of the Voronoi cell containing $q(t)$.

Our goal is to prove that $s(t)$ belongs to a ball of radius $\frac{\sqrt{3}}{2}||a-b||$ centered at c (Lemma 6.13) and that this ball contains at most $2(k+3)$ sample points. We begin with the latter.

Lemma 6.10. *For $(a,b) \in G_k$ and c as defined above, the ball B' of radius $\frac{\sqrt{3}}{2}||a-b||$ centered at c contains at most $2(k+3)$ sample points.*

Proof. Consider the ball B_a with center a and radius $||a-b||$. Every sample point in the interior of this ball is closer to a than b is. Thus, B_a has at most k points in its interior. Also B_a has at most four points in its boundary by our non-degeneracy assumption. Similarly, the ball B_b with center b and radius $||a-b||$ also contains at most $k+4$ points.

The ball B' is completely contained in the union of B_a and B_b and thus it contains at most $2(k+4)-2$ points. The -2 accounts for the fact that a and b are contained in both balls. □

Next we estimate the distance from c to $s(t)$.

Lemma 6.11. $||c-s(t)|| \leq ||a-b||/2 + 2||p(t)-q(t)||$

Proof. Assume w.l.o.g that $||a-p(t)|| \leq ||b-p(t)||$, otherwise we do the computation with b. By the triangle inequality, $||c-s(t)|| \leq ||c-p(t)|| + ||p(t)-q(t)|| + ||q(t)-s(t)||$. Since $q(t)$ is closer to $s(t)$ than any other sample point, $||q(t)-s(t)|| \leq ||q(t)-a||$. Moreover, $||q(t)-a|| \leq ||q(t)-p(t)|| + ||p(t)-a||$. Finally, $||a-p(t)|| + ||p(t)-c|| = ||a-b||/2$. □

It remains to bound $||p(t)-q(t)||$ as a function of $||a-b||$. We first estimate the distance of $q(t)$ from the medial axis.

Lemma 6.12. $f(q(t)) > (\lambda-1)||a-b||$

Proof. Assume w.l.o.g that $||a-p(t)|| \leq ||b-p(t)||$, otherwise we do the computation with b. Since $q(t)$ is the point in S closest to $p(t)$, we have $||a-q(t)|| \leq ||a-p(t)|| + ||p(t)-q(t)|| \leq 2||a-p(t)|| \leq ||a-b||$. By Lemma 6.2 $f(q(t)) \geq f(a) - ||a-q(t)||$ and hence $f(q(t)) \geq f(a) - ||a-b|| > (\lambda-1)||a-b||$. □

Lemma 6.13. *For $\lambda \geq 2$, $||p(t)-q(t)|| \leq \frac{\sqrt{3}-1}{4}||a-b||$ and $||c-s(t)|| \leq \frac{\sqrt{3}}{2}||a-b||$.*

Proof. Consider the point $q(t)$. By Lemma 6.12 there are two medial balls with radius at least $(\lambda-1)||a-b||$ tangent to $q(t)$. The surface passes between these balls and does not intersect their interior, in particular, a and b do not lie in the interior of these balls. Thus, the worst case (when the distance

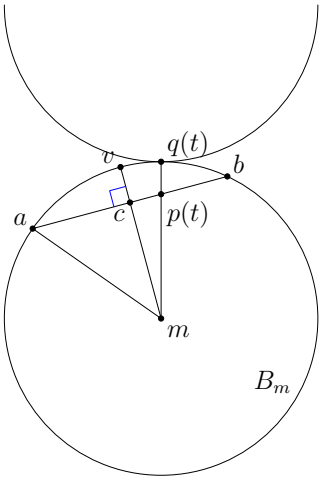


Figure 6.2: Bounding $||p(t) - q(t)||$ in terms of $||a - b||$.

$||p(t) - q(t)||$ compared to $||a - b||$ is maximized) occurs, when both lie on the boundary of one of these balls (see Figure 6.2). Let m be the center of this ball and use B_m to denote the ball. Consider the perpendicular bisector of segment ab passing through m. It intersects segment ab at c and ball B_m at v. Also, $p(t)$ is on the segment $q(t)m$.

Distance $||p(t) - q(t)||$ is upper bounded by $||c - v||$ and thus we are left with bounding $||c - v||$. Referring to Figure 6.2 we see that the triangle acm is right. Thus, $||c - m||^2 = ||a - m||^2 - ||a - c||^2$. Moreover,

$$||a - m|| = ||v - m|| = \zeta||a - b|| \text{ for some } \zeta \geq \lambda - 1 \ ,$$

and $||a - c|| = ||a - b||/2$. Combining all these,

$$||c - m|| = \sqrt{\zeta^2 - \frac{1}{4}} \cdot ||a - b|| \ .$$

Finally, $||c - v|| = \zeta||a - b|| - ||c - m||$ and hence

$$\begin{aligned} ||p(t) - q(t)|| \leq ||c - v|| &= \left(\zeta - \sqrt{\zeta^2 - \frac{1}{4}}\right) ||a - b|| \\ &\leq (1 - \sqrt{\frac{3}{4}})||a - b|| \\ &< \frac{\sqrt{3} - 1}{4}||a - b|| \ . \end{aligned}$$

This proves the first part of the lemma. The second part follows now from

Lemma 6.11 since,

$$||c - s(t)|| \leq \frac{1}{2}||a - b|| + 2||p(t) - q(t)|| \leq \frac{\sqrt{3}}{2}||a - b|| \ . \qquad \square$$

It is now easy to complete the proof of Theorem 6.6. Set $\lambda = 2$. Then, p_{ab} is contained in the ball B' and B' contains at most $2(k + 3)$ sample points. Thus, p_{ab} has length at most $2k + 5$. Together with the edge e we get a cycle of length at most $2(k + 3)$.

Recall that our goal is to satisfy the assumptions of Section 6.2.1. If we combine Theorem 6.6 and Lemma 6.8 we get that the edges of G_k are small in length.

Corollary 6.14. *Let $\alpha = (1 + \epsilon)/(1 - \epsilon)$. For any edge $e = (a, b) \in G_k$,*

$$||a - b|| \leq (\alpha^{2k+5} - 1) \min(f(a), f(b)) \ . \tag{6.3}$$

Moreover, we will need the following lemma which can be easily derived from [38, Lemma 10 and Theorem 4].

Lemma 6.15 (Giesen and Wagner [38])**.** *Let a and b be two points of S such that $||a - b|| \leq \eta \cdot \min(f(a), f(b))$ with $\eta \leq 1/4$. Then, $L(\gamma_{ab}) \leq 4||a - b||$ where $L(\gamma_{ab})$ denotes the length of γ_{ab}.*

Using Lemma 6.15 we set c_1 to 4 in Section 6.2.1.

6.2.4 Long Cycles

In this section we make precise how non-trivial cycles are long. The idea is simple; non-trivial cycles have a certain minimum length and edges of G_k are short. We will actually prove a stronger result. The length of a non-trivial loop is bounded from below by the maximum feature size of any point on the loop. Combined with Lemma 6.15, we will obtain the desired result.

Assume that $\eta = \alpha^{2k+5} - 1 \leq \frac{1}{4}$. Let C be any non-trivial cycle of G_k. Substituting each edge $(a, b) \in C$ by the curve γ_{ab} gives us a non-trivial loop γ on S. By Lemma 6.15 and Corollary 6.14 we get

$$L(\gamma) \leq 4 \sum_{(a,b)\in C} ||a - b|| \leq 4 \sum_{(a,b)\in C} \eta \min(f(a), f(b)) \ ,$$

and if $|C|$ denotes the number of edges of C,

$$L(\gamma) \leq 4\eta|C| \max_{a\in C} f(a) \ . \tag{6.4}$$

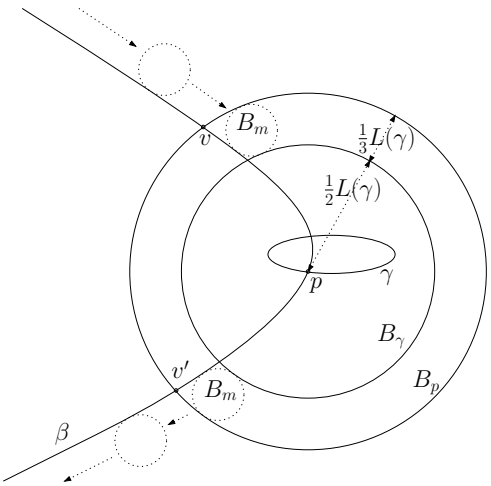


Figure 6.3: Proving Theorem 6.16.

In order to get a lower bound on $|C|$ we need to relate γ to its distance to the medial axis. More precisely, we are going to show the following theorem which might be of independent interest.

Theorem 6.16. *Let γ be any non-trivial loop on S, then $L(\gamma) \geq \max_{p \in \gamma} f(p)$.*

Proof. Let p be a point on γ with maximum distance from the medial axis and assume $L(\gamma) < f(p)$ for the sake of contradiction. Let β be a non-trivial loop of different homology class going through p (see Figure 6.3). At each point $x \in \beta$ there are two tangent balls with radius $f(x)$ which do not contain any point of S in their interior. One of these tangent balls when moving it along β (and adjusting its size accordingly) produces an object $\mathcal{T}$, topologically equivalent to a torus, around which γ loops non-trivially.

Let B_p be the ball with center p and radius $5L(\gamma)/6$. For all $x \in B_p \cap \beta$, the local feature size is large, namely, $f(x) \geq f(p) - ||p - x|| > L(\gamma) - 5L(\gamma)/6 = L(\gamma)/6$ and hence for $x \in B_p \cap \beta$ the ball defining $\mathcal{T}$ has radius at least $L(\gamma)/6$.

The loop γ stays inside a ball of radius $L(\gamma)/2$ centered at p and hence well inside B_p. Since γ loops around $\mathcal{T}$ its length it least $2\pi L(\gamma)/6 > L(\gamma)$, a contradiction. □

We can now establish that non-trivial cycles in G_k are long.

Theorem 6.17. *For appropriate values of ϵ, δ, and k any non-trivial cycle $C \in G_k$ has length $|C| \geq \frac{1}{4\eta}$ where $\eta = \alpha^{2k+5} - 1 \leq \frac{1}{4}$.*

ϵ	valid k	chosen k	η	short-cycles upper bound	long-cycles lower bound
10^{-2}	$[15, 20]$	15	1.014	36	-
5×10^{-3}	$[14, 40]$	14	0.391	34	-
10^{-3}	$[14, 202]$	14	0.068	34	4
5×10^{-4}	$[14, 405]$	14	0.033	34	8
10^{-4}	$[14, 2027]$	14	0.0066	34	38
5×10^{-5}	$[14, 4054]$	14	0.0033	34	76

Table 6.1: Evaluation of the various conditions for the separation of the MCB *for different values of the sampling density ϵ.*

Proof. Using inequality (6.4) and Theorem 6.16 we obtain

$$L(\gamma) \leq 4\eta|C| \max_{a \in C} f(a) \leq 4\eta|C|L(\gamma) \ ,$$

and the theorem follows. □

Corollary 6.18. *If $\eta = \alpha^{2k+5} - 1 < \frac{1}{16(k+3)}$ then all non-trivial cycles in G_k have length larger than $4(k+3)$.*

Proof. We fix k and ϵ to some constants according (a) to our assumptions in Section 6.2.3 and (b) such that $\eta < \frac{1}{16(k+3)}$. Then, by Theorem 6.17 $|C| \geq \frac{1}{4\eta} > 4(k+3)$. □

Putting everything together establishes Theorem 6.1.

Corollary 6.19. *If the conditions in Section 6.2.1 hold, the sampling density is high enough and k is large enough: every MCB of G_k contains exactly $m-(n-1)-2g$ short (length less than $2(k+3)$) and exactly $2g$ long (length at least $4(k+3)$) cycles.*

Proof. Use $D_{P|S}$ as M in the assumptions of Section 6.2.1. By Lemma 6.15 we can set $c_1 = 4$. Theorem 6.6, Corollary 6.14 and Theorem 6.17 prove the remaining assumptions. The corollary follows by the proof of Theorem 6.1 for $c_2 = 3$ and $c_3 = 2k+5$. □

6.2.5 Putting It All Together

Our assumptions so far suggest that given an (ϵ, δ)-sample and $w = \alpha - 1 = \frac{2\epsilon}{1-\epsilon}$, we should choose k such that:

$$\frac{(\delta(1+w)+w)^2}{\delta^2(1-w)^2 - w^4} \leq k < \log_{\frac{1+\epsilon}{1-\epsilon}} \frac{3}{2} \ . \tag{6.5}$$

There are values of ϵ such that inequality (6.5) cannot be satisfied. However, as ϵ decreases the right hand side increases while the left hand side decreases. Thus, both conditions can always be simultaneously satisfied for some dense enough sample. The above conditions are what is required for the trivial cycles of the MCB to have length at most $2(k+3)$.

For the lower bound on the length of the non-trivial cycles of the MCB, we also require that $\eta = \alpha^{2k+5} - 1 \leq \frac{1}{4}$, and in order for the length of the non-trivial cycles to reach the desired number $|C| \geq \frac{1}{4\eta} > 4(k+3)$ we require that $\eta = \alpha^{2k+5} - 1 < \frac{1}{16(k+3)}$.

We fix δ to $3w/8 \approx 3\epsilon/4$ and evaluate, in Table 6.1, the bounds for different values of ϵ.

Remark The lower and upper bounds presented in Table 6.1 are not tight. Somewhat large constants appear due to the proof technique used. Perhaps, by using some other proximity graph instead of the k-neighborhood or by performing a different analysis the bounds can be improved. This is especially true for the value of ϵ required by the theory in order for the non-trivial cycles to be longer than the trivial ones. In practice this is true for smaller values of k and sampling densities. The next section presents experimental data which confirm this.

6.3 Experimental Validation

6.3.1 Genus Determination

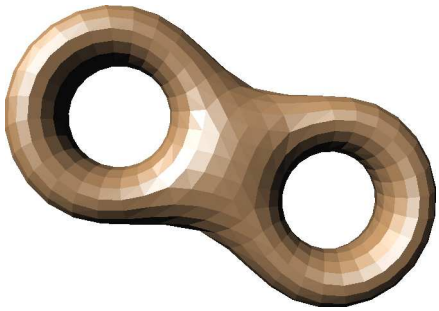
This section presents experimental data on the size of trivial and non-trivial cycles in the MCBs of point clouds sampled from compact manifolds. The main observation is that the MCB cycles are separated into the two categories, short trivial and long non-trivial, for rather small values of k and sampling density. Moreover, the upper bound on the length of the trivial cycles is much less than $2(k+3)$ and the method also works for some non-smooth samples.

We study four different examples:

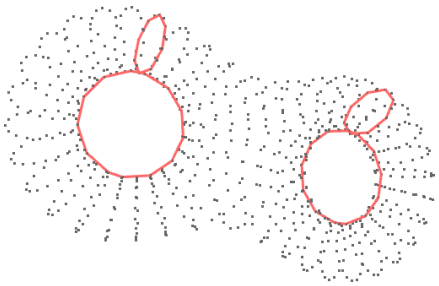
(a) the "double torus", a genus 2 double torus (Figure 6.4a) with a sparse point cloud (Figure 6.4b),

(b) the "bumpy torus", a genus 1 surface (Figure 6.4c) with a dense point cloud (Figure 6.4d),

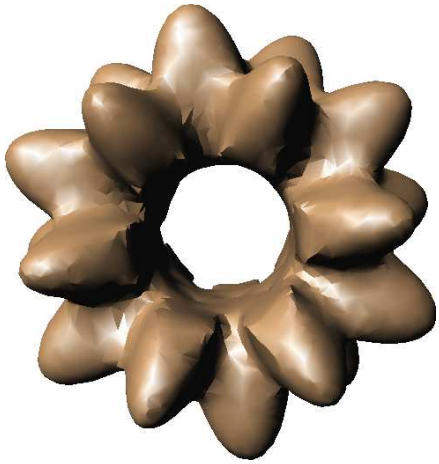
(c) the "homer torus", a genus 1 surface (Figure 6.5a) with a sparse point cloud (Figure 6.5b), and



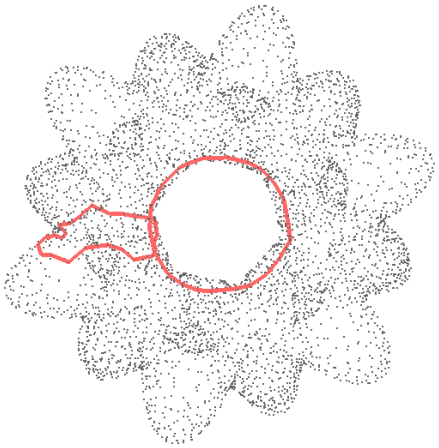
(a) Double torus



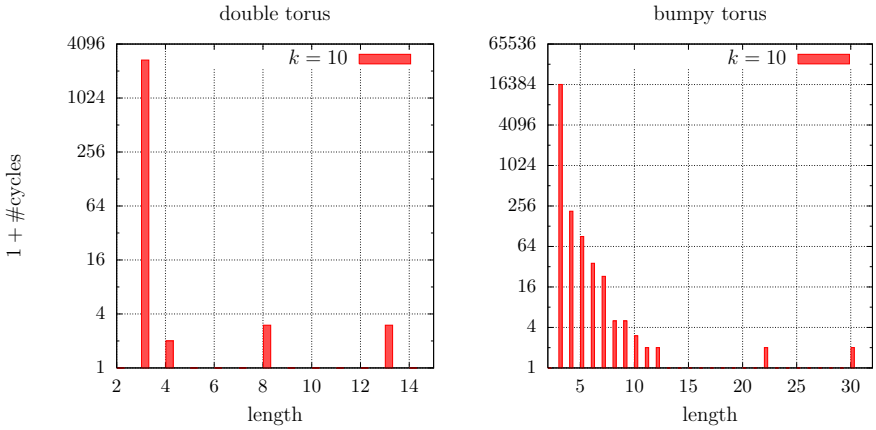
(b) 767 points cloud



(c) Bumpy torus



(d) 5044 points cloud



(e) Distribution of the lengths of the MCB cycles of G_k of the double and bumpy torus models for $k = 10$.

Figure 6.4: The "double" and "bumpy" torus models. The red cycles are the long non-trivial cycles extracted from the MCB of G_k for $k = 10$.

(d) the "wedge torus", a genus 1 non-smooth surface (Figure 6.6a) with a sparse point cloud (Figure 6.6b).

Double torus

Since the model has genus 2 we expect the MCB to reveal exactly 4 non-trivial cycles. The minimum value of k for this event to happen is 6. All cycles but four of the MCB have length at most 8, two cycles have length 13 and two 24. As k increases this gap grows, for $k = 8$ two cycles have length 11, two others have length 24 and all the rest at most 5. For $k = 10$ two cycles have length 8, two 13 and the rest at most 4. This continues to be true as we increase k as long as the edges of the k-nearest neighbor graph do not shortcut a handle.

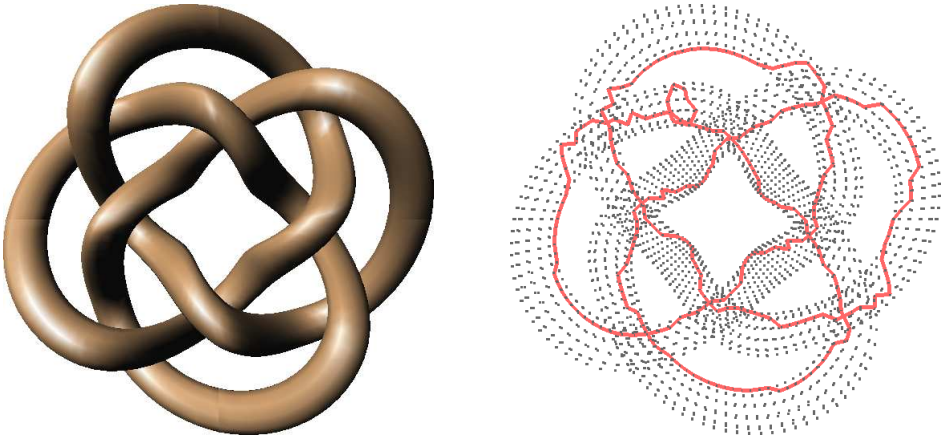
Although the proof of Lemma 6.10 fails in the case of the *unsymmetric* k-nearest neighbor graph, where an edge is added even if only one endpoint is a k-nearest neighbor of the other, in practice we observe the same behavior. The values of k are even smaller for this case. For minimum $k = 5$ there are two cycles of length 11, two of length 24 and the rest at most 6. For $k = 9$ all MCB cycles are triangles except two with length 9 and two with length 14.

Bumpy torus

The situation improves if the sampling is dense. The "bumpy torus" model has genus one and thus we expect the MCB to reveal two non-trivial cycles. For $k = 6$ the two non-trivial cycles have length 35 and 42 and the rest at most 27. Due to the density of the sample as we increase k this difference becomes more noticeable. For $k = 10$ the non-trivial cycles have length 22 and 30 and the rest at most 12. For $k = 12$ the two non-trivial have length 20 and 26 and the rest at most 9. Note also that in all these examples almost all trivial cycles have length 3 or 4. For example, when $k = 12$ about 99% of the trivial cycles are triangles. See Figure 6.4e for a histogram of the cycles lengths when $k = 10$.

Homer torus

This model has again genus 1 and thus we expect the MCB to reveal two non-trivial cycles. For $k = 7$ the MCB contains one cycle of length 252, one of length 8 and all the remaining have length at most 5. Our point cloud however is considerably sparse and thus it is quite easy for the MCB to fail



(a) Homer torus *(b) 3486 points cloud*

Figure 6.5: The "homer" torus model and the two non-trivial cycles revealed by the MCB for $k = 7$. The two non-trivial cycles have length 8 and 252 respectively.

to detect the two non-trivial cycles for other values of k. We elaborate more on this at the end of this section.

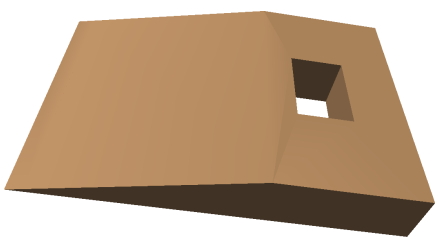
Wedge torus

The non-smooth surface has genus one. We expect that as long as k is not too large, our method should reveal two non-trivial cycles. Figure 6.6b shows the two non-trivial cycles of the MCB for $k = 22$. Note that this is a difficult instance for surface reconstruction.

Discussion

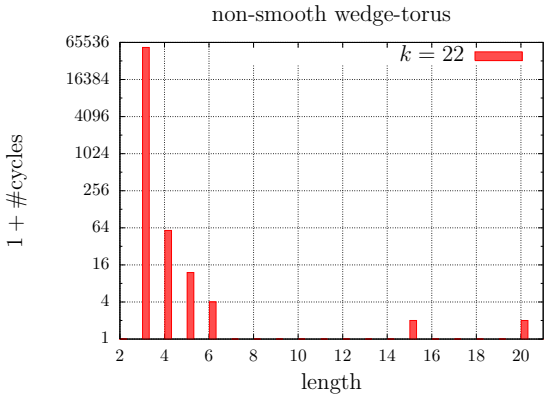
In the examples above, the MCB was able to reveal the genus. There were exactly $2g$ long cycles and long and short cycles are clearly discernible by length. However, in practice, if g is unknown and the sampling density not as high as it should be, how can g be determined from the MCB? We suggest the following heuristic. Let l be the minimal integer such that the MCB contains no cycle of length between l and $2l$ inclusive. Then, $2g$ is the number of cycles of length larger than l. If this number is odd, this is an indication of insufficient sampling density or a wrong value of k.

What can go wrong when the sample is not sufficiently dense or the value of k is not chosen properly? When k is too small, the MCB might contain long trivial cycles. When k is too large, G_k may contain edges between



(a) Wedge torus

(b) 6078 points cloud



(c) Distribution of the lengths of the MCB cycles for $k = 22$.

Figure 6.6: The "wedge" torus model and the two non-trivial cycles revealed by the MCB for $k = 22$.

points distant from each other in S and hence spurious long cycles may enter the basis, see Figure 6.7. The figure also shows that non-smoothness by itself is not an obstacle.

6.4 Application to Surface Reconstruction

In this section we outline the approach of Tewari et al. [75] for surface reconstruction. The interested reader is referred to their paper for more details.

Tewari et al. [75] show that if a basis for the trivial loops of the manifold may be computed from the sample of a 2-manifold of genus 1, it is possible to parameterize the sample set, and then construct a piecewise-linear approximation to the surface. They use the MCB of the k-nearest neighbor graph to extract this basis, assuming that the non-trivial cycles are the two longest ones. They observed that this is correct if the sample is dense

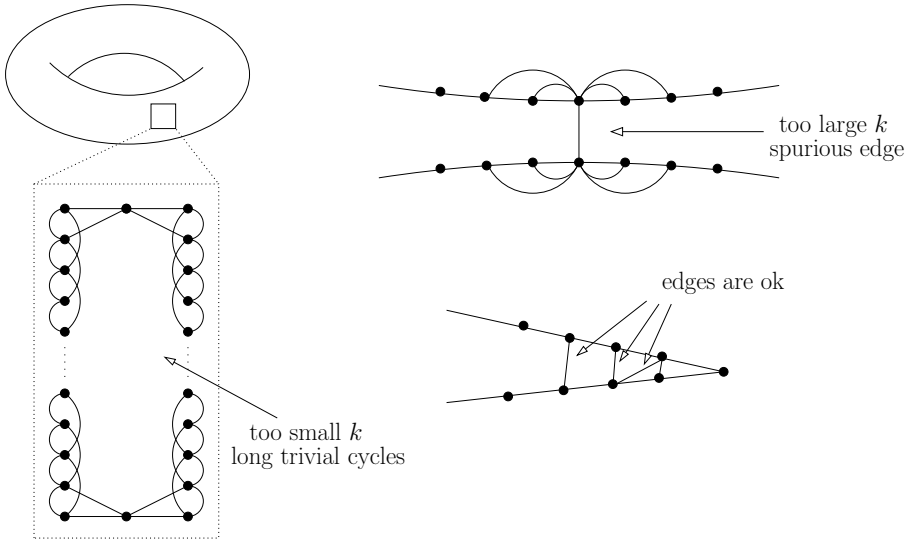


Figure 6.7: Several difficult situations. In the left figure due to symmetry a small choice for k ($k = 4$) leads to long trivial cycles. In the upper right figure a large value of k results in an edge which connects two parts of the surface which are distant from each other. The lower right figure shows that edges near non-smooth features are not a problem as long as k is not too large.

enough, but did not prove anything in this respect. Theorem 6.1 shows under which conditions this reconstruction algorithm provably constructs a triangulation homeomorphic to the surface.

The parameterization based approach has its origins in Tutte's "spring embedder" for planar graphs [79]. Tutte introduced a simple, yet powerful method for producing straight-line drawings of planar graphs. The vertices of the outer face are mapped to the vertices of a convex polygon and all other vertices are placed at the centroid of their neighbors. Algorithmically, this amounts to solving a linear system of equations. If we use p_v to denote the location of vertex v and N_v to denote the set of neighbors of v, this means

$$p_v = \sum_{w \in N_v} \lambda_{vw} p_w \quad \text{and} \quad \lambda_{vw} = 1/|N_v| \tag{6.6}$$

for every interior vertex v. Tutte proved that the coordinates computed in this way define a non-degenerate embedding for any 3-connected planar graph. Floater [32] showed that the result stays true if vertices are placed

at arbitrary convex combinations of their neighbors, i.e.,

$$\sum_{w \in N_v} \lambda_{vw} = 1 \quad \text{and} \quad \lambda_{vw} \geq 0 \ .$$

For the sequel, it is convenient to rewrite Equation (6.6) as

$$\sum_{w \in N_v} \lambda_{vw}(p_w - p_v) = 0$$

and to introduce x_{vw} for the vector from v to w in the embedding. Gortler et al. [41] extended the method to embeddings onto the torus. Given a 3-connected map (= graph + cyclic ordering on the edges incident to any vertex) of genus one, they viewed undirected edges $\{v, w\}$ as pairs of directed edges and associated a variable x_{vw} with every directed edge. They used the equations:

$$\begin{aligned}
x_{vw} + x_{wv} &= 0 && \text{for all edges } (v, w) && \text{(symmetry)} \\
\sum_{w \in N_v} \lambda_{vw} x_{vw} &= 0 && \text{for all vertices } v && \text{(center of gravity)} \\
\sum_{(w,v) \in \partial f} x_{wv} &= 0 && \text{for all faces } f && \text{(face sums)}
\end{aligned}$$

The first class of equations ensures that the vector from w to v is the same as the vector from v to w, the second class ensures that v is a convex combination of its neighbors, and the third class ensures that the vectors along any face boundary sum to zero. There are $2m$ unknowns and $m+n+f$ equations. Two equations are redundant (one center of gravity constraint and one face sum constraint) and hence the rank of the system is $m + n + f - 2 = 2m - 2$ (this uses the Euler theorem $f - m + n = 2 - 2g = 0$). Gortler et al., extending results of Gu and Yau [44], proved that two independent solutions can be used as the x and y-coordinates of an embedding onto a torus.

Floater and Reimers [33] observed that Tutte's method can also be used to reconstruct surfaces with boundary of genus zero and Tewari et al. [75] extended the observation to closed surfaces of genus one, as follows. Construct the k-nearest neighbor graph G_k of P and then set up the equations introduced above. Face sum constraints are needed for a basis of the trivial cycles and this is exactly what an MCB yields. The solution of the system defines an embedding of P onto the torus. A triangulation, say the Delaunay triangulation, of the embedded point set is computed and then lifted

back to the original point set. In this way a genus 1 surface interpolating P is obtained. The surface may have self-intersections. Postprocessing can be used to improve the quality of the mesh.

6.5 Concluding Remarks

In this chapter we have shown that given a suitably nice sample of a smooth manifold of genus g and sufficiently large k, the k-nearest neighbor graph of the sample has a cycle basis consisting only of *short* (= length at most $2(k+3)$) and *long* (= length at least $4(k+3)$) cycles. Moreover, the MCB is such a basis and contains exactly $m-(n-1)-2g$ short cycles and $2g$ long cycles. The short cycles span the subspace of trivial loops and the long cycles form a homology basis. Thus, the MCB reveals the genus of S and also provides a basis for the set of trivial cycles and a set of generators for the non-trivial cycles of S. These cycles may then be used to parameterize P and ultimately generate a piecewise linear manifold surface approximating S.

In our experiments we observe that the length separation of trivial and non-trivial cycles happens already for relatively sparse samples. In addition, this threshold is less than $2(k+3)$. Furthermore, our experiments suggest that the method also works for some non-smooth surfaces.

As a final remark we note that a constant factor approximate MCB has similar properties as the MCB for sufficiently dense samples.

Chapter 7

Conclusions

This thesis studies the minimum cycle basis problem in undirected graphs. Our work can be logically divided into four parts. The nature of the first three is algorithmic while the fourth is structural. In the first part we improve existing algorithms with respect to the running time, reaching an $O(m^2n + mn^2 \log n)$ bound in the general case. For special cases of undirected graphs we improve the bounds even further. In the second part we approximate the minimum cycle basis problem and provide time bounds which for sufficiently dense graphs are $o(m^\omega)$. In the third part we experiment with various implementations of MCB algorithms. We design several heuristics, from simple to more complex, which improve the running times dramatically. In the fourth part of this thesis, we study properties of the minimum cycle bases of neighborhood graphs of point clouds sampled from compact smooth 2-manifolds. Such MCBs encode topological and geometrical information about the underlying manifold. We note that most of the techniques developed in this thesis are extendible in the directed minimum cycle basis problem where the base field is no longer $\mathbb{F}_2$ but $\mathbb{Q}$.

There are still quite a few remaining questions regarding minimum cycle bases in undirected graphs. First of all, we only presented improved upper bounds for finding an MCB. An interesting open problem is to derive a lower bound on the running time of any MCB algorithm for an undirected graph. The approach that we used in Chapter 3 is very unlikely to be improved to something better that $\Theta(m^\omega)$. Is therefore something like an $\Omega(m^\omega)$ lower bound for the MCB problem?

The second main question is whether Algorithm 3.3 can be further improved and reach an $O(m^\omega)$ upper bound even for sparse graphs. In order to accomplish this we need to reduce the time taken by the shortest paths

computations. One way to accomplish this is to find the shortest cycle with an odd number of edges from some subset of the edges $S \subseteq E$ in time $o(\text{APSP}(n, m))$ where $\text{APSP}(n, m)$ is the time to perform all pairs shortest paths in an undirected graph with n vertices and m edges.

From a practical perspective we are interested in computing sparse cycle bases very fast. A constant factor approximate cycle basis is such a sparse cycle basis. One of the most important open question in this area is to decouple the cycle basis computation from the null space basis. We have partially solved this by computing all but $O(n^{1+1/k})$ cycles without using a null space basis.

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Notation

$G(V, E)$	Graph G with vertex set V and edge set E
V or $V(G)$	Vertex set of graph G
E or $E(G)$	Edge set of graph G
n	Cardinality of the vertex set of a graph
m	Cardinality of the edge set of a graph
$\kappa(G)$	Number of weakly connected components of graph G
N	Dimension of cycle space of G, $N = m - n + \kappa(G)$
(u, v)	Edge between vertices u and v
$\deg(v)$	The degree of vertex v
$\delta(u, v)$	The shortest path weight from u to v
$\mathcal{V}(G)$	Vertex space of graph G
$\mathcal{E}(G)$	Edge space of graph G
$\mathcal{C}(G)$	Cycle space of graph G
$A \oplus B$	$(A \setminus B) \cup (B \setminus A)$
$\tilde{O}(f(n))$	$O(f(n) \operatorname{poly}(\log n))$
$\langle a, b \rangle$	Inner product of vectors a and b
$[n]$	Set $\{1, 2, \ldots, n\}$
$G \cong G'$	Groups G and G' are isomorphic

$C_k(M;R)$	Chain group
∂_k	Boundary operator
Z_k	Subgroup of k-cycles in C_k
B_k	Subgroup of k-boundaries in C_k
$H_k(M;R)$	k-th homology group
G_k	k-nearest neighbor graph
$f : S \mapsto \mathbb{R}$	local feature size (least distance to the medial axis of S)
D_P	Delaunay triangulation of point sample P
V_P	Voronoi diagram of point sample P
$D_{P\|S}$	Restricted Delaunay triangulation of point sample P to surface S
$V_{P\|S}$	Restricted Voronoi diagram of point sample P to surface S
γ_{ab}	Continuous curve on a manifold connecting a and b
$\|\|a-b\|\|$	Euclidean distance between vectors a and b

Zusammenfassung

Wir betrachten das Problem, eine minimale Kreisbasis eines ungerichteten, Kanten-gewichteten Graphen G mit m Kanten und n Knoten zu berechnen. Hierbei assoziieren wir mit jedem Kreis einen $\{0,1\}$ Inzidenzvektor. Der Vektorraum über $\mathbb{F}_2$, der durch diese Vektoren erzeugt wird, heißt Kreisraum von G. Eine Menge von Kreisen wird als Kreisbasis von G bezeichnet, wenn sie eine Basis des Kreisraumes von G ist. Eine Kreisbasis heißt minimal, wenn die Summe der Gewichte ihrer Kreise minimal ist. Minimale Kreisbasen werden in vielen Kontexten verwendet, z.B. in der Analyse von elektrischen Netzwerken und in der Chemie.

Wir präsentieren einen Algorithmus mit Laufzeit $O(m^2n+mn^2 \log n)$ um eine solche minimale Kreisbasis zu berechnen. Die beste bekannte Laufzeit war $O(m^\omega n)$, wobei ω der Exponent der Matrix-Multiplikation ist. Zurzeit bekannt ist, daß $\omega < 2.376$ gilt. Wenn die Kantengewichte ganzzahlig sind, geben wir einen $O(m^2n)$ Algorithmus an. Für ungewichtete Graphen, die hinreichend dicht sind, läuft unser Algorithmus in $O(m^\omega)$ Zeit.

Weiterhin entwickeln wir Approximationsalgorithmen für das Minimale Kreisbasen Problem. Für jedes $\epsilon > 0$ entwickeln wir einen schnellen $(1+\epsilon)$-Approximations Algorithmus. Außerdem, präsentieren wir für jede ganze Zahl $k \geq 1$ zwei Approximationsalgorithmen, die beide einen Approximationsfaktor von $2k-1$ haben. Der eine hat erwartete Laufzeit $O(kmn^{1+2/k} + mn^{(1+1/k)(\omega-1)})$ und der andere Laufzeit $O(n^{3+2/k})$. Für Graphen, die hinreichend dicht sind, sind die Laufzeiten $o(m^\omega)$. Für spezielle Graphen, wie planare oder geometrische, präsentieren wir noch schnellere Approximationsalgorithmen. Unsere Methoden sind auch auf gerichtete Graphen anwendbar.

Wir untersuchen das Minimale Kreisbasis Problem ebenfalls aus einer praktischen Perspektive. Wir beschreiben, wie man den $O(m^3 + mn^2 \log n)$ Algorithmus von de Pina effizient implementiert. Wir entwickeln verschiedene

Heuristiken, die die Laufzeit beträchtlich verbessern. Indem wir die zwei fundamental verschiedenen Methoden, eine minimale Kreisbasis zu berechnen, kombinieren, erhalten wir einen $O(m^2n^2)$ Algorithmus. Weiterhin vergleichen wir verschiedene Algorithmen anhand von Experimenten.

Schließlich untersuchen wir die minimale Kreisbasis eines „nearest neighbor“ Graphen, der auf eine Stichprobenmenge einer Oberfläche im $\mathbb{R}^3$ definiert wird. Wir zeigen, dass unter bestimmten Bedingungen die minimale Kreisbasis topologische Informationen über die Oberfläche kodiert und Basen der trivialen und nicht-trivialen Kreise der Oberfläche liefert. Wir bestätigen unsere Ergebnisse anhand von Experimente.

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